

CK-12

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CHAPTER 5



CHAPTER 5**Relationships with
Triangles****Chapter Outline**

- 5.1 MIDSEGMENTS OF A TRIANGLE**
 - 5.2 PERPENDICULAR BISECTORS IN TRIANGLES**
 - 5.3 ANGLE BISECTORS IN TRIANGLES**
 - 5.4 MEDIANS AND ALTITUDES IN TRIANGLES**
 - 5.5 INEQUALITIES IN TRIANGLES**
 - 5.6 EXTENSION: INDIRECT PROOF**
 - 5.7 CHAPTER 5 REVIEW**
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This chapter introduces different segments within triangles and how they relate to each other. We will explore the properties of midsegments, perpendicular bisectors, angle bisectors, medians, and altitudes. Next, we will look at the relationship of the sides of a triangle, how they relate to each other and how the sides of one triangle can compare to another.

5.1 Midsegments of a Triangle

Learning Objectives

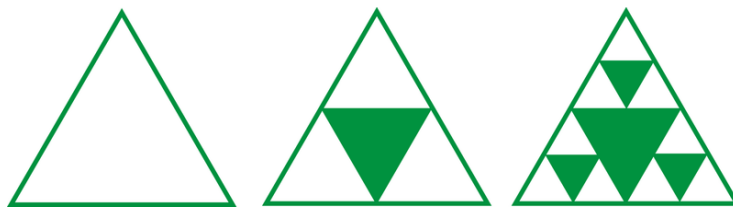
- Identify the midsegments of a triangle.
- Use the Midsegment Theorem to solve problems involving side lengths, midsegments, and algebra.

Review Queue

Find the midpoint between the given points.

- $(-4, 1)$ and $(6, 7)$
- $(5, -3)$ and $(11, 5)$
- $(0, -2)$ and $(-4, 14)$
- Find the equation of the line between $(-2, -3)$ and $(-1, 1)$.
- Find the equation of the line that is parallel to the line from #4 through $(2, -7)$.

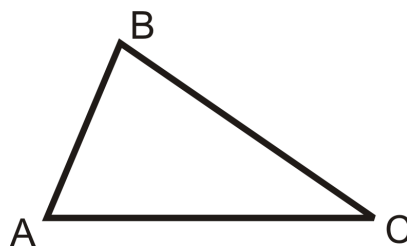
Know What? A fractal is a repeated design using the same shape (or shapes) of different sizes. Below, is an example of the first few steps of a fractal. What does the next figure look like? How many triangles are in each figure (green and white triangles)? Is there a pattern?



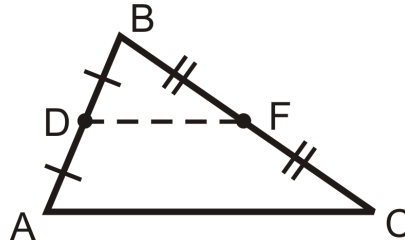
Defining Midsegment

Midsegment: A line segment that connects two midpoints of adjacent sides of a triangle.

Example 1: Draw the midsegment \overline{DF} between \overline{AB} and \overline{BC} . Use appropriate tic marks.

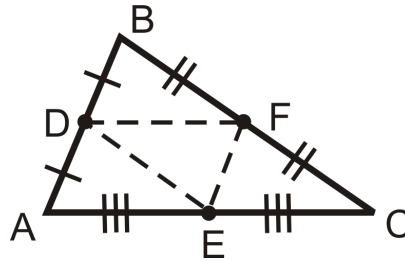


Solution: Find the midpoints of \overline{AB} and \overline{BC} using your ruler. Label these points D and F . Connect them to create the midsegment.



Don't forget to put the tic marks, indicating that D and F are midpoints, $\overline{AD} \cong \overline{DB}$ and $\overline{BF} \cong \overline{FC}$.

Example 2: Find the midpoint of \overline{AC} from $\triangle ABC$. Label it E and find the other two midsegments of the triangle.



Solution:

For every triangle there are three midsegments.

Let's transfer what we know about midpoints in the coordinate plane to midsegments in the coordinate plane. We will need to use the midpoint formula, $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

Example 3: The vertices of $\triangle LMN$ are $L(4,5)$, $M(-2,-7)$ and $N(-8,3)$. Find the midpoints of all three sides, label them O , P and Q . Then, graph the triangle, it's midpoints and draw in the midsegments.

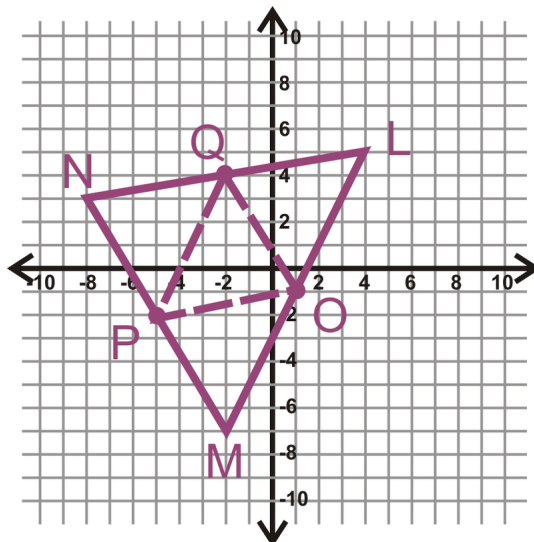
Solution: Use the midpoint formula 3 times to find all the midpoints.

$$L \text{ and } M = \left(\frac{4+(-2)}{2}, \frac{5+(-7)}{2} \right) = (1, -1), \text{ point } O$$

$$L \text{ and } N = \left(\frac{4+(-8)}{2}, \frac{5+3}{2} \right) = (-2, 4), \text{ point } Q$$

$$M \text{ and } N = \left(\frac{-2+(-8)}{2}, \frac{-7+3}{2} \right) = (-5, -2), \text{ point } P$$

The graph would look like the graph to the right. We will use this graph to explore the properties of midsegments.



Example 4: Find the slopes of \overline{NM} and \overline{QO} .

Solution: The slope of \overline{NM} is $\frac{-7-3}{-2-(-8)} = \frac{-10}{6} = -\frac{5}{3}$.

The slope of \overline{QO} is $\frac{-1-4}{1-(-2)} = -\frac{5}{3}$.

From this we can conclude that $\overline{NM} \parallel \overline{QO}$. If we were to find the slopes of the other sides and midsegments, we would find $\overline{LM} \parallel \overline{QP}$ and $\overline{NL} \parallel \overline{PO}$. **This is a property of all midsegments.**

Example 5: Find NM and QO .

Solution: Now, we need to find the lengths of \overline{NM} and \overline{QO} . Use the distance formula.

$$NM = \sqrt{(-7-3)^2 + (-2-(-8))^2} = \sqrt{(-10)^2 + 6^2} = \sqrt{100+36} = \sqrt{136} \approx 11.66$$

$$QO = \sqrt{(1-(-2))^2 + (-1-4)^2} = \sqrt{3^2 + (-5)^2} = \sqrt{9+25} = \sqrt{34} \approx 5.83$$

From this we can conclude that QO is **half** of NM . If we were to find the lengths of the other sides and midsegments, we would find that OP is **half** of NL and QP is **half** of LM . **This is a property of all midsegments.**

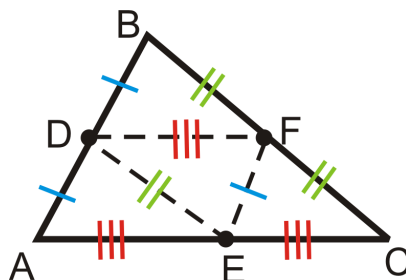
The Midsegment Theorem

The conclusions drawn in Examples 4 and 5 can be generalized into the Midsegment Theorem.

Midsegment Theorem: The midsegment of a triangle is half the length of the side it is parallel to.

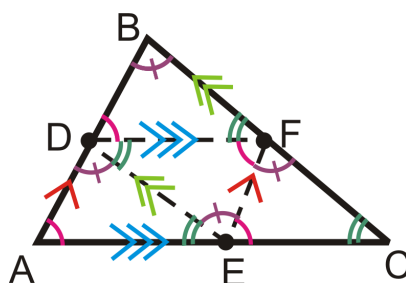
Example 6: Mark everything you have learned from the Midsegment Theorem on $\triangle ABC$ above.

Solution: Let's draw two different triangles, one for the congruent sides, and one for the parallel lines.



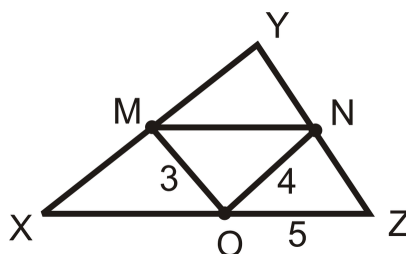
Because the midsegments are half the length of the sides they are parallel to, they are congruent to half of each of those sides (as marked). Also, this means that all four of the triangles in $\triangle ABC$, created by the midsegments are congruent by SSS.

As for the parallel midsegments and sides, several congruent angles are formed. In the picture to the right, the pink and teal angles are congruent because they are corresponding or alternate interior angles. Then, the purple angles are congruent by the 3rd Angle Theorem.



To play with the properties of midsegments, go to <http://www.mathopenref.com/trianglemidsegment.html> .

Example 7: $M, N,$ and O are the midpoints of the sides of the triangle.



Find

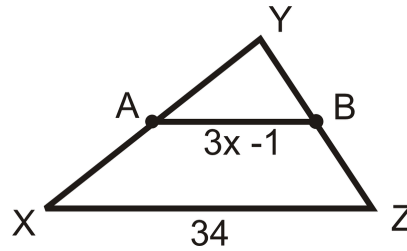
- a) MN
- b) XY
- c) The perimeter of $\triangle XYZ$

Solution: Use the Midsegment Theorem.

- a) $MN = OZ = 5$
- b) $XY = 2(ON) = 2 \cdot 4 = 8$
- c) The perimeter is the sum of the three sides of $\triangle XYZ$.

$$XY + YZ + XZ = 2 \cdot 4 + 2 \cdot 3 + 2 \cdot 5 = 8 + 6 + 10 = 24$$

Example 8: Algebra Connection Find the value of x and AB .



Solution: First, AB is half of 34, or 17. To find x , set $3x - 1$ equal to 17.

$$3x - 1 = 17$$

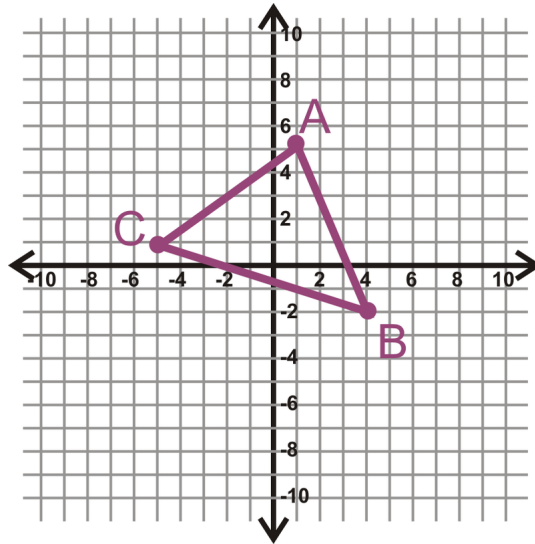
$$3x = 18$$

$$x = 6$$

Let's go back to the coordinate plane.

Example 9: If the midpoints of the sides of a triangle are $A(1, 5)$, $B(4, -2)$, and $C(-5, 1)$, find the vertices of the triangle.

Solution: The easiest way to solve this problem is to graph the midpoints and then apply what we know from the Midpoint Theorem.



Now that the points are plotted, find the slopes between all three.

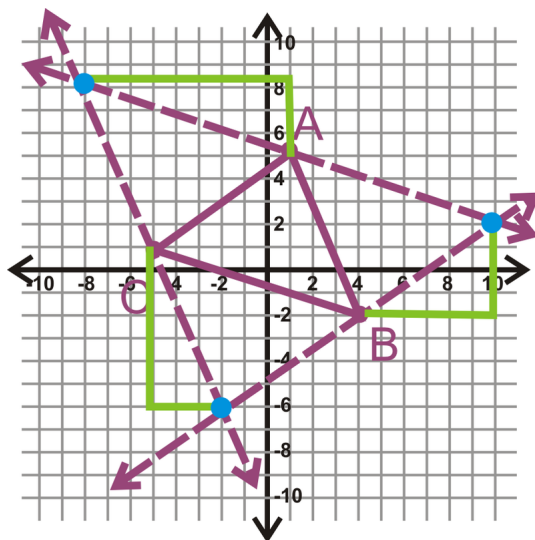
$$\text{slope } AB = \frac{5-2}{1-4} = -\frac{7}{3}$$

$$\text{slope } BC = \frac{-2-1}{4+5} = \frac{-3}{9} = -\frac{1}{3}$$

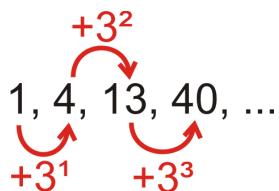
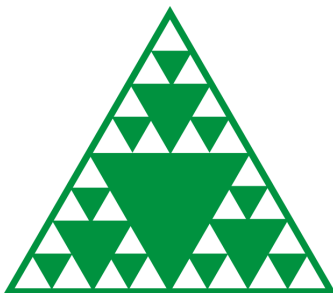
$$\text{slope } AC = \frac{5-1}{1+5} = \frac{4}{6} = \frac{2}{3}$$

Using the slope between two of the points and the third, plot the slope triangle on either side of the third point and extend the line. Repeat this process for all three midpoints. For example, use the slope of AB with point C .

The green lines in the graph to the left represent the slope triangles of each midsegment. The three dotted lines represent the sides of the triangle. Where they intersect are the vertices of the triangle (the blue points), which are $(-8, 8)$, $(10, 2)$ and $(-2, 6)$.

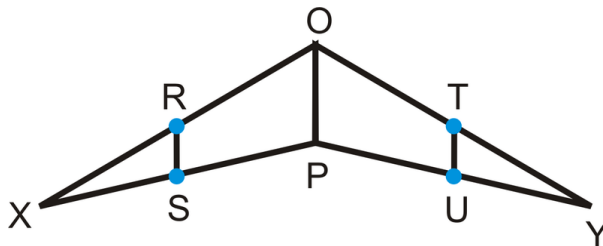


Know What? Revisited To the left is a picture of the 4th figure in the fractal pattern. The number of triangles in each figure is 1, 4, 13, and 40. The pattern is that each term increase by the next power of 3.



Review Questions

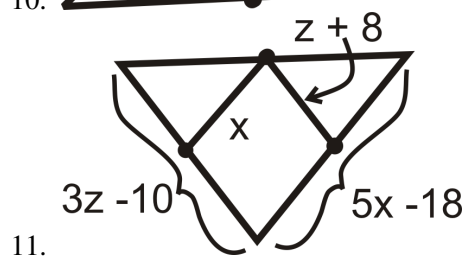
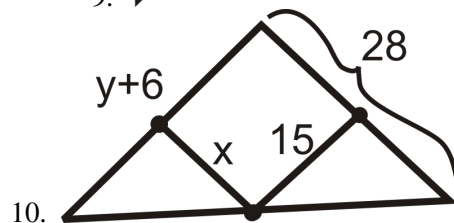
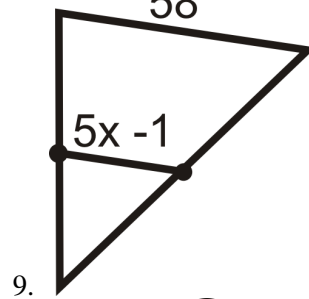
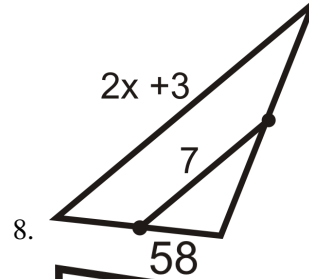
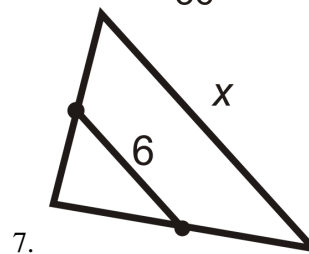
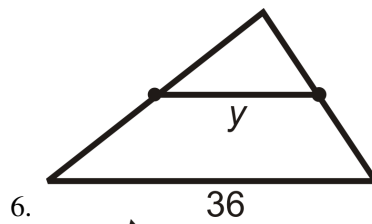
$R, S, T,$ and U are midpoints of the sides of $\triangle XPO$ and $\triangle YPO$.

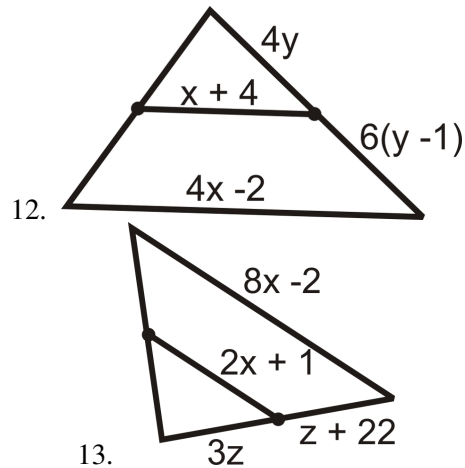


1. If $OP = 12$, find RS and TU .
2. If $RS = 8$, find TU .

3. If $RS = 2x$, and $OP = 20$, find x and TU .
4. If $OP = 4x$ and $RS = 6x - 8$, find x .
5. Is $\triangle XOP \cong \triangle YOP$? Why or why not?

For questions 6-13, find the indicated variable(s). You may assume that all line segments within a triangle are midsegments.





14. The sides of $\triangle XYZ$ are 26, 38, and 42. $\triangle ABC$ is formed by joining the midpoints of $\triangle XYZ$.
- Find the perimeter of $\triangle ABC$.
 - Find the perimeter of $\triangle XYZ$.
 - What is the relationship between the perimeter of a triangle and the perimeter of the triangle formed by connecting its midpoints?

Coordinate Geometry Given the vertices of $\triangle ABC$ below, find the midpoints of each side.

- $A(5, -2)$, $B(9, 4)$ and $C(-3, 8)$
- $A(-10, 1)$, $B(4, 11)$ and $C(0, -7)$
- $A(0, 5)$, $B(4, -1)$ and $C(-2, -3)$
- $A(2, 4)$, $B(8, -4)$ and $C(2, -4)$

Multi-Step Problem The midpoints of the sides of $\triangle RST$ are $G(0, -2)$, $H(9, 1)$, and $I(6, -5)$. Answer the following questions.

- Find the slope of \overline{GH} , \overline{HI} , and \overline{GI} .
- Plot the three midpoints and connect them to form midsegment triangle, $\triangle GHI$.
- Using the slopes, find the coordinates of the vertices of $\triangle RST$.
- Find \overline{GH} using the distance formula. Then, find the length of the sides it is parallel to. What should happen?

More Coordinate Geometry Given the midpoints of the sides of a triangle, find the vertices of the triangle. Refer back to problems 19-21 for help.

- $(-2, 1)$, $(0, -1)$ and $(-2, -3)$
- $(1, 4)$, $(4, 1)$ and $(2, 1)$

Given the vertices of $\triangle ABC$, find:

- the midpoints of \overline{MN} and \overline{NO} where M is the midpoint of \overline{AB} , N is the midpoint of \overline{BC} and O is the midpoint of \overline{AC} .
- Show that midsegments \overline{MN} , \overline{NO} and \overline{OM} are parallel to sides \overline{AC} , \overline{AB} and \overline{BC} respectively.
- Show that midsegments \overline{MN} , \overline{NO} and \overline{OM} are half the length of sides \overline{AC} , \overline{AB} and \overline{BC} respectively.

- $A(-3, 5)$, $B(3, 1)$ and $C(-5, -5)$
- $A(-2, 2)$, $B(4, 4)$ and $C(6, 0)$

For questions 27-30, $\triangle CAT$ has vertices $C(x_1, y_1)$, $A(x_2, y_2)$ and $T(x_3, y_3)$.

27. Find the midpoints of sides \overline{CA} and \overline{CT} . Label them L and M respectively.
28. Find the slopes of \overline{LM} and \overline{AT} .
29. Find the lengths of \overline{LM} and \overline{AT} .
30. What have you just proven algebraically?

Review Queue Answers

- a. $\left(\frac{-4+6}{2}, \frac{1+7}{2}\right) = (1, 4)$
- b. $\left(\frac{5+11}{2}, \frac{-3+5}{2}\right) = (8, 1)$
- c. $\left(\frac{0-4}{2}, \frac{-2+14}{2}\right) = (-2, 6)$
- d. $m = \frac{-3-1}{-2-(-1)} = \frac{-4}{-1} = 4$
 $y = mx + b$
 $-3 = 4(-2) + b$
 $b = 5, y = 4x + 5$
- e. $-7 = 4(2) + b$
 $b = -15, y = 4x - 15$

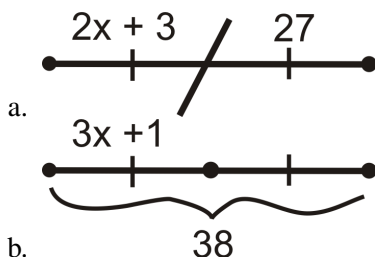
5.2 Perpendicular Bisectors in Triangles

Learning Objectives

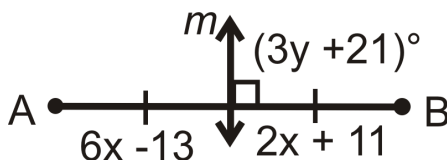
- Understand points of concurrency.
- Apply the Perpendicular Bisector Theorem and its converse to triangles.
- Understand concurrency for perpendicular bisectors.

Review Queue

- Construct the perpendicular bisector of a 3 inch line. Use Investigation 1-3 from Chapter 1 to help you.
- Find the value of x .

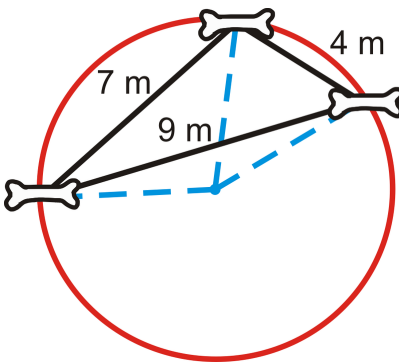


- Find the value of x and y . Is m the perpendicular bisector of AB ? How do you know?



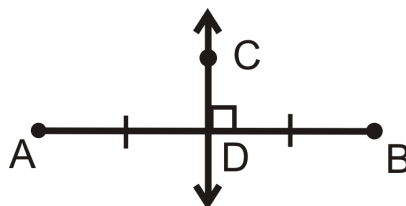
Know What? An archeologist has found three bones in Cairo, Egypt. The bones are 4 meters apart, 7 meters apart and 9 meters apart (to form a triangle). The likelihood that more bones are in this area is very high. The archeologist wants to dig in an appropriate circle around these bones. If these bones are on the edge of the digging circle, where is the center of the circle?

Can you determine how far apart each bone is from the center of the circle? What is this length?



Perpendicular Bisectors

In Chapter 1, you learned that a perpendicular bisector intersects a line segment at its midpoint and is perpendicular. In #1 in the Review Queue above, you constructed a perpendicular bisector of a 3 inch segment. Let's analyze this figure.



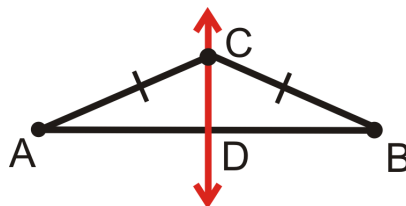
\overleftrightarrow{CD} is the perpendicular bisector of \overline{AB} . If we were to draw in \overline{AC} and \overline{CB} , we would find that they are equal. Therefore, any point on the perpendicular bisector of a segment is the same distance from each endpoint.

Perpendicular Bisector Theorem: If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

The proof of the Perpendicular Bisector Theorem is in the exercises for this section. In addition to the Perpendicular Bisector Theorem, we also know that its converse is true.

Perpendicular Bisector Theorem Converse: If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Proof of the Perpendicular Bisector Theorem Converse



Given: $\overline{AC} \cong \overline{CB}$

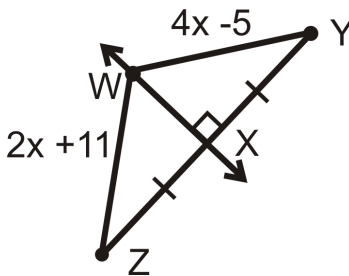
Prove: \overleftrightarrow{CD} is the perpendicular bisector of \overline{AB}

TABLE 5.1:

<i>Statement</i>	<i>Reason</i>
1. $\overline{AC} \cong \overline{CB}$	Given
2. $\triangle ACB$ is an isosceles triangle	Definition of an isosceles triangle
3. $\angle A \cong \angle B$	Isosceles Triangle Theorem
4. Draw point D , such that D is the midpoint of \overline{AB} .	Every line segment has exactly one midpoint
5. $\overline{AD} \cong \overline{DB}$	Definition of a midpoint
6. $\triangle ACD \cong \triangle BCD$	SAS
7. $\angle CDA \cong \angle CDB$	CPCTC
8. $m\angle CDA = m\angle CDB = 90^\circ$	Congruent Supplements Theorem
9. $\overleftrightarrow{CD} \perp \overline{AB}$	Definition of perpendicular lines
10. \overleftrightarrow{CD} is the perpendicular bisector of \overline{AB}	Definition of perpendicular bisector

Let's use the Perpendicular Bisector Theorem and its converse in a few examples.

Example 1: Algebra Connection Find x and the length of each segment.

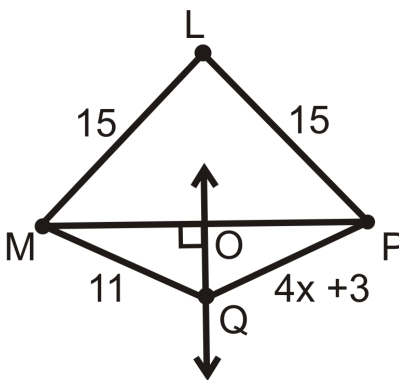


Solution: From the markings, we know that \overleftrightarrow{WX} is the perpendicular bisector of \overline{WY} . Therefore, we can use the Perpendicular Bisector Theorem to conclude that $WZ = WY$. Write an equation.

$$\begin{aligned} 2x + 11 &= 4x - 5 \\ 16 &= 2x \\ 8 &= x \end{aligned}$$

To find the length of WZ and WY , substitute 8 into either expression, $2(8) + 11 = 16 + 11 = 27$.

Example 2: \overleftrightarrow{OQ} is the perpendicular bisector of \overline{MP} .



- a) Which segments are equal?
- b) Find x .
- c) Is L on \overleftrightarrow{OQ} ? How do you know?

Solution:

- a) $ML = LP$ because they are both 15.
 $MO = OP$ because O is the midpoint of \overline{MP}
 $MQ = QP$ because Q is on the perpendicular bisector of \overline{MP} .

$$\begin{aligned} \text{b) } 4x + 3 &= 11 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

- c) Yes, L is on \overleftrightarrow{OQ} because $ML = LP$ (Perpendicular Bisector Theorem Converse).

Perpendicular Bisectors and Triangles

Two lines intersect at a point. If more than two lines intersect at the same point, it is called a point of concurrency.

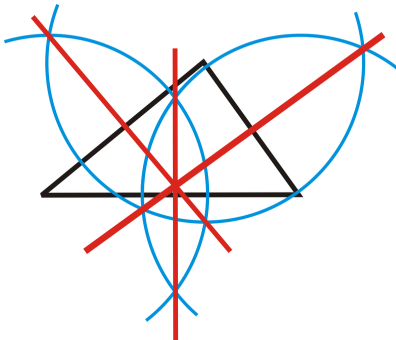
Point of Concurrency: When three or more lines intersect at the same point.

Investigation 5-1: Constructing the Perpendicular Bisectors of the Sides of a Triangle

Tools Needed: paper, pencil, compass, ruler

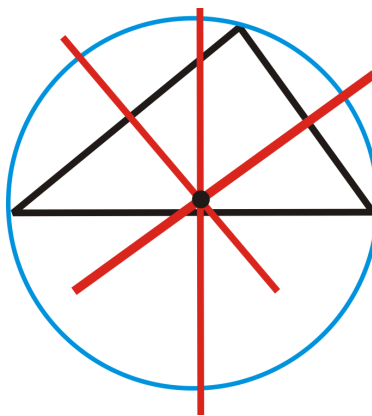
1. Draw a scalene triangle.
2. Construct the perpendicular bisector (Investigation 1-3) for all three sides.

The three perpendicular bisectors all intersect at the same point, called the circumcenter.

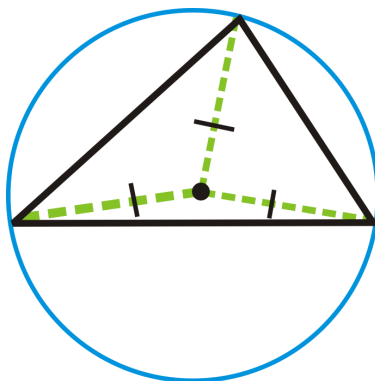


Circumcenter: The point of concurrency for the perpendicular bisectors of the sides of a triangle.

3. Erase the arc marks to leave only the perpendicular bisectors. Put the pointer of your compass on the circumcenter. Open the compass so that the pencil is on one of the vertices. Draw a circle. What happens?

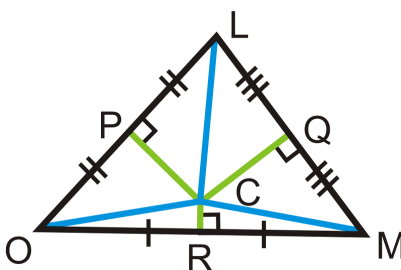


The circumcenter is the center of a circle that passes through all the vertices of the triangle. We say that this circle *circumscribes* the triangle. This means that *the circumcenter is equidistant to the vertices*.



Concurrency of Perpendicular Bisectors Theorem: The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the vertices.

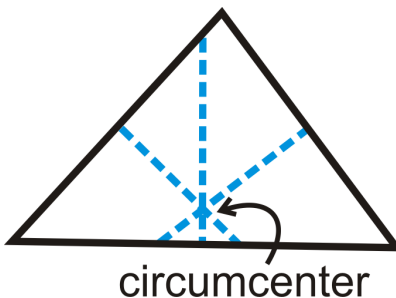
If \overline{PC} , \overline{QC} , and \overline{RC} are perpendicular bisectors, then $LC = MC = OC$.



Example 3: For further exploration, try the following:

- Cut out an acute triangle from a sheet of paper.
- Fold the triangle over one side so that the side is folded in half. Crease.
- Repeat for the other two sides. What do you notice?

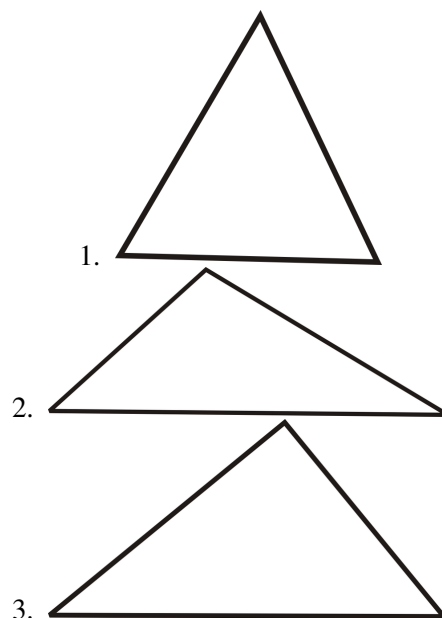
Solution: The folds (blue dashed lines) are the perpendicular bisectors and cross at the circumcenter.



Know What? Revisited The center of the circle will be the circumcenter of the triangle formed by the three bones. Construct the perpendicular bisector of at least two sides to find the circumcenter. After locating the circumcenter, the archeologist can measure the distance from each bone to it, which would be the radius of the circle. This length is approximately 4.7 meters.

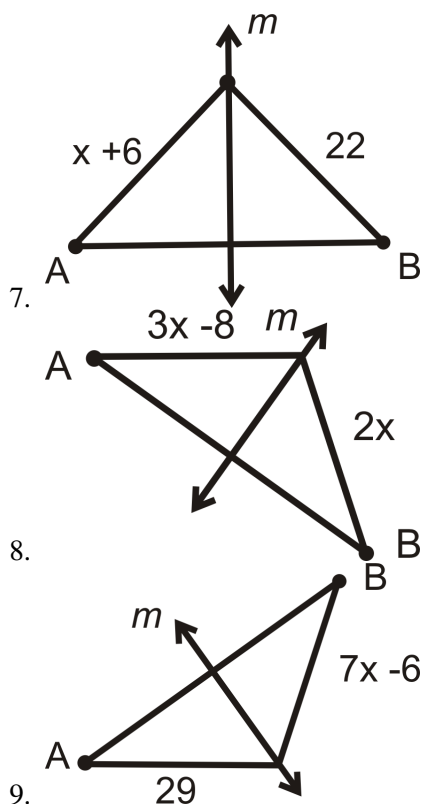
Review Questions

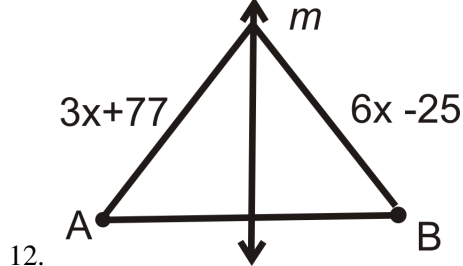
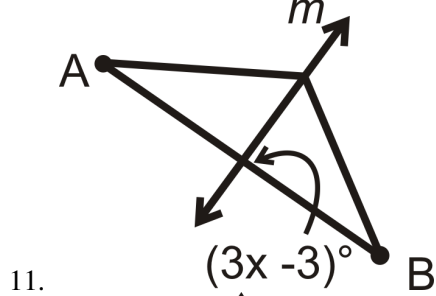
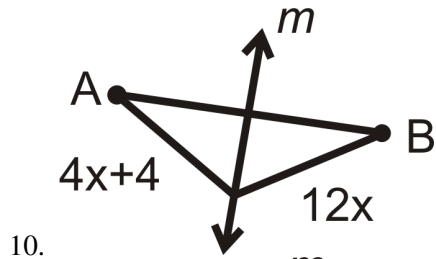
Construction Construct the circumcenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-1.



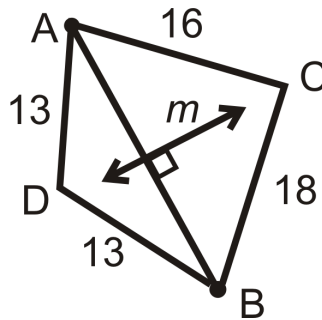
4. Can you use the method in Example 3 to locate the circumcenter for these three triangles?
5. Based on your constructions in 1-3, state a conjecture about the relationship between a triangle and the location of its circumcenter.
6. Construct equilateral triangle $\triangle ABC$ (Investigation 4-6). Construct the perpendicular bisectors of the sides of the triangle and label the circumcenter X . Connect the circumcenter to each vertex. Your original triangle is now divided into six triangles. What can you conclude about the six triangles? Why?

Algebra Connection For questions 7-12, find the value of x . m is the perpendicular bisector of AB .

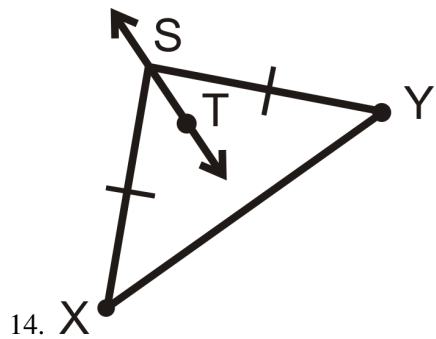


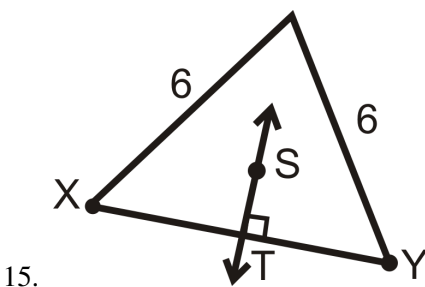


13. m is the perpendicular bisector of \overline{AB} .
- List all the congruent segments.
 - Is C on \overline{AB} ? Why or why not?
 - Is D on \overline{AB} ? Why or why not?



For Questions 14 and 15, determine if \overleftrightarrow{ST} is the perpendicular bisector of \overline{XY} . Explain why or why not.



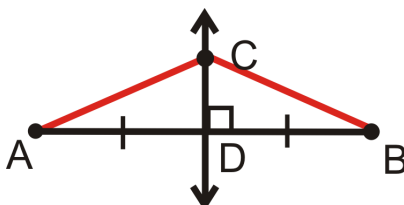


For Questions 16-20, consider line segment \overline{AB} with endpoints $A(2, 1)$ and $B(6, 3)$.

16. Find the slope of AB .
17. Find the midpoint of AB .
18. Find the equation of the perpendicular bisector of AB .
19. Find AB . Simplify the radical, if needed.
20. Plot A, B , and the perpendicular bisector. Label it m . How could you find a point C on m , such that C would be the third vertex of equilateral triangle $\triangle ABC$? *You do not have to find the coordinates, just describe how you would do it.*

For Questions 21-25, consider $\triangle ABC$ with vertices $A(7, 6), B(7, -2)$ and $C(0, 5)$. Plot this triangle on graph paper.

21. Find the midpoint and slope of \overline{AB} and use them to draw the perpendicular bisector of \overline{AB} . You do not need to write the equation.
22. Find the midpoint and slope of \overline{BC} and use them to draw the perpendicular bisector of \overline{BC} . You do not need to write the equation.
23. Find the midpoint and slope of \overline{AC} and use them to draw the perpendicular bisector of \overline{AC} . You do not need to write the equation.
24. Are the three lines concurrent? What are the coordinates of their point of intersection (what is the circumcenter of the triangle)?
25. Use your compass to draw the circumscribed circle about the triangle with your point found in question 24 as the center of your circle.
26. Repeat questions 21-25 with $\triangle LMO$ where $L(2, 9), M(3, 0)$ and $O(-7, 0)$.
27. Repeat questions 21-25 with $\triangle REX$ where $R(4, 2), E(6, 0)$ and $X(0, 0)$.
28. Can you explain why the perpendicular bisectors of the sides of a triangle would all pass through the center of the circle containing the vertices of the triangle? Think about the definition of a circle: The set of all point equidistant from a given center.
29. Fill in the blanks: There is exactly _____ circle which contains any _____ points.
30. Fill in the blanks of the proof of the Perpendicular Bisector Theorem.



Given: \overleftrightarrow{CD} is the perpendicular bisector of \overline{AB} Prove: $\overline{AC} \cong \overline{CB}$

TABLE 5.2:

Statement

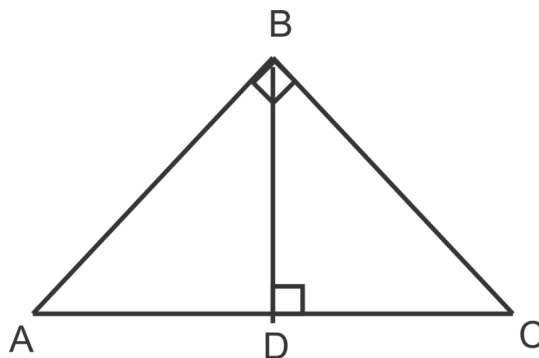
Reason

1.

TABLE 5.2: (continued)

Statement	Reason
2. D is the midpoint of \overline{AB}	
3.	Definition of a midpoint
4. $\angle CDA$ and $\angle CDB$ are right angles	
5. $\angle CDA \cong \angle CDB$	
6.	Reflexive PoC
7. $\triangle CDA \cong \triangle CDB$	
8. $\overline{AC} \cong \overline{CB}$	

31. Write a two column proof. Given: $\triangle ABC$ is a right isosceles triangle and \overline{BD} is the \perp bisector of \overline{AC} Prove: $\triangle ABD$ and $\triangle CBD$ are congruent.



32. Write a paragraph explaining why the two smaller triangles in question 31 are also isosceles right triangles.

Review Queue Answers

- a. Reference Investigation 1-3.

$$\begin{aligned} \text{a. } 2x + 3 &= 27 \\ 2x &= 24 \\ x &= 12 \end{aligned}$$

$$\begin{aligned} \text{b. } 3x + 1 &= 19 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

$$\begin{aligned} \text{b. } 6x - 13 &= 2x + 11 & 3y + 21 &= 90^\circ \\ 4x &= 24 & 3y &= 69^\circ \\ x &= 6 & y &= 23^\circ \end{aligned}$$

Yes, m is the perpendicular bisector of AB because it is perpendicular to AB and passes through the midpoint.

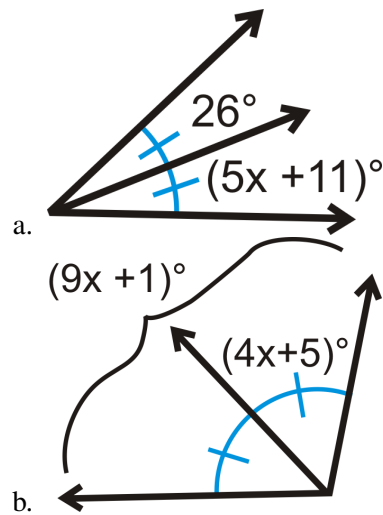
5.3 Angle Bisectors in Triangles

Learning Objectives

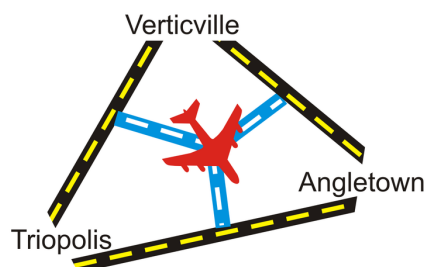
- Apply the Angle Bisector Theorem and its converse.
- Understand concurrency for angle bisectors.

Review Queue

- Construct the angle bisector of an 80° angle (Investigation 1-4).
- Draw the following: M is on the interior of $\angle LNO$. O is on the interior of $\angle MNP$. If \overrightarrow{NM} and \overrightarrow{NO} are the angle bisectors of $\angle LNO$ and $\angle MNP$ respectively, write all the congruent angles.
- Find the value of x .



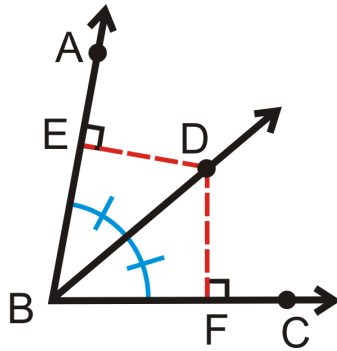
Know What? The cities of Verticville, Triopolis, and Angletown are joining their city budgets together to build a centrally located airport. There are freeways between the three cities and they want to have the freeway on the interior of these freeways. Where is the best location to put the airport so that they have to build the least amount of road?



In the picture to the right, the blue roads are proposed.

Angle Bisectors

In Chapter 1, you learned that an angle bisector cuts an angle exactly in half. In #1 in the Review Queue above, you constructed an angle bisector of an 80° angle. Let's analyze this figure.



\overrightarrow{BD} is the angle bisector of $\angle ABC$. Looking at point D , if we were to draw \overline{ED} and \overline{DF} , we would find that they are equal. Recall from Chapter 3 that the shortest distance from a point to a line is the perpendicular length between them. ED and DF are the shortest lengths between D , which is on the angle bisector, and each side of the angle.

Angle Bisector Theorem: If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

In other words, if \overrightarrow{BD} bisects $\angle ABC$, $\overrightarrow{BE} \perp \overline{ED}$, and $\overrightarrow{BF} \perp \overline{DF}$, then $ED = DF$.

Proof of the Angle Bisector Theorem

Given: \overrightarrow{BD} bisects $\angle ABC$, $\overrightarrow{BA} \perp \overline{AD}$, and $\overrightarrow{BC} \perp \overline{DC}$

Prove: $\overline{AD} \cong \overline{DC}$

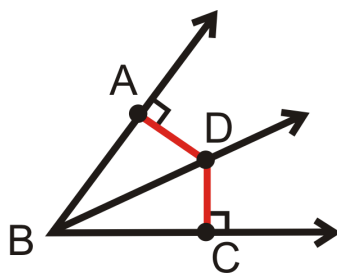


TABLE 5.3:

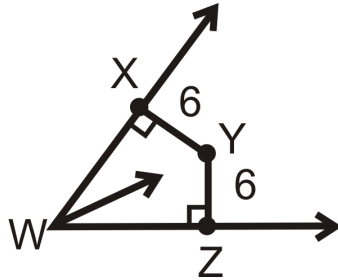
Statement	Reason
1. \overrightarrow{BD} bisects $\angle ABC$, $\overrightarrow{BA} \perp \overline{AD}$, $\overrightarrow{BC} \perp \overline{DC}$	Given
2. $\angle ABD \cong \angle DBC$	Definition of an angle bisector
3. $\angle DAB$ and $\angle DCB$ are right angles	Definition of perpendicular lines
4. $\angle DAB \cong \angle DCB$	All right angles are congruent
5. $\overline{BD} \cong \overline{BD}$	Reflexive PoC
6. $\triangle ABD \cong \triangle CBD$	AAS
7. $\overline{AD} \cong \overline{DC}$	CPCTC

The converse of this theorem is also true. The proof is in the review questions.

Angle Bisector Theorem Converse: If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

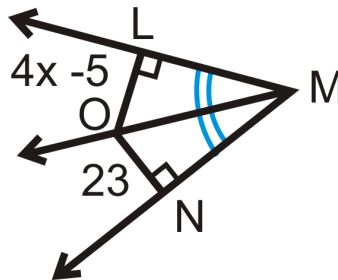
Because the Angle Bisector Theorem and its converse are both true we have a biconditional statement. We can put the two conditional statements together using if and only if. *A point is on the angle bisector of an angle if and only if it is equidistant from the sides of the triangle.*

Example 1: Is Y on the angle bisector of $\angle XWZ$?



Solution: In order for Y to be on the angle bisector XY needs to be equal to YZ and they both need to be perpendicular to the sides of the angle. From the markings we know $\overline{XY} \perp \overline{WX}$ and $\overline{ZY} \perp \overline{WZ}$. Second, $XY = YZ = 6$. From this we can conclude that Y is on the angle bisector.

Example 2: \overrightarrow{MO} is the angle bisector of $\angle LMN$. Find the measure of x .



Solution: $LO = ON$ by the Angle Bisector Theorem Converse.

$$4x - 5 = 23$$

$$4x = 28$$

$$x = 7$$

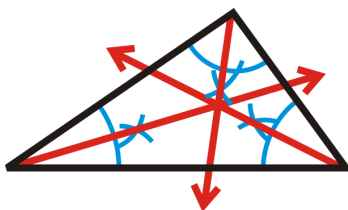
Angle Bisectors in a Triangle

Like perpendicular bisectors, the point of concurrency for angle bisectors has interesting properties.

Investigation 5-2: Constructing Angle Bisectors in Triangles

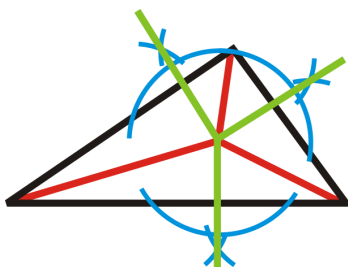
Tools Needed: compass, ruler, pencil, paper

1. Draw a scalene triangle. Construct the angle bisector of each angle. Use Investigation 1-4 and #1 from the Review Queue to help you.



Incenter: The point of concurrency for the angle bisectors of a triangle.

2. Erase the arc marks and the angle bisectors after the incenter. Draw or construct the perpendicular lines to each side, through the incenter.



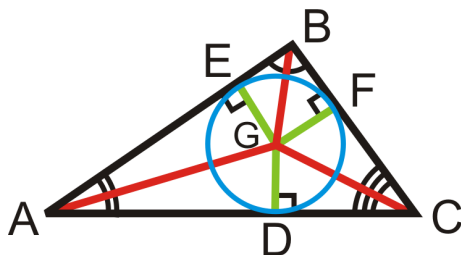
3. Erase the arc marks from #2 and the perpendicular lines beyond the sides of the triangle. Place the pointer of the compass on the incenter. Open the compass to intersect one of the three perpendicular lines drawn in #2. Draw a circle.



Notice that the circle touches all three sides of the triangle. We say that this circle is *inscribed* in the triangle because it touches all three sides. The incenter is on all three angle bisectors, so *the incenter is equidistant from all three sides of the triangle*.

Concurrency of Angle Bisectors Theorem: The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

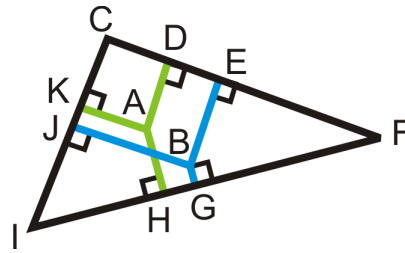
If \overline{AG} , \overline{BG} , and \overline{CG} are the angle bisectors of the angles in the triangle, then $EG = GF = GD$.



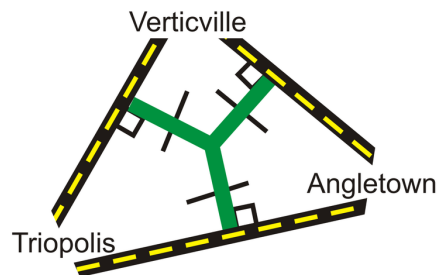
In other words, \overline{EG} , \overline{FG} , and \overline{DG} are the radii of the inscribed circle.

Example 3: If J , E , and G are midpoints and $KA = AD = AH$ what are points A and B called?

Solution: A is the incenter because $KA = AD = AH$, which means that it is equidistant to the sides. B is the circumcenter because \overline{JB} , \overline{BE} , and \overline{BG} are the perpendicular bisectors to the sides.

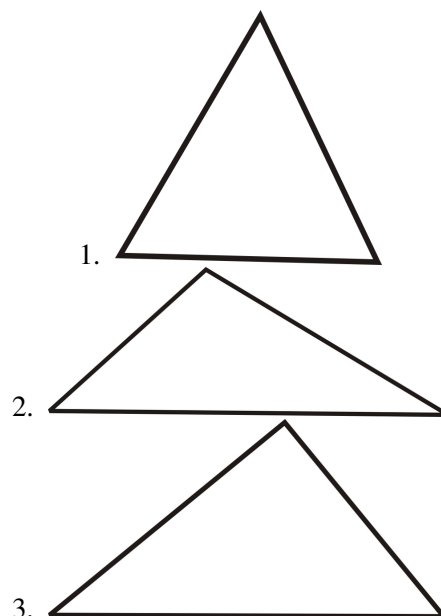


Know What? Revisited The airport needs to be equidistant to the three highways between the three cities. Therefore, the roads are all perpendicular to each side and congruent. The airport should be located at the incenter of the triangle.



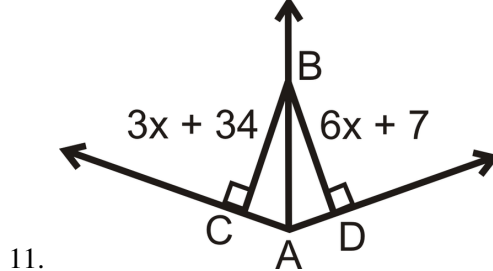
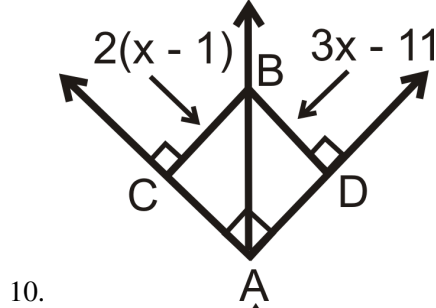
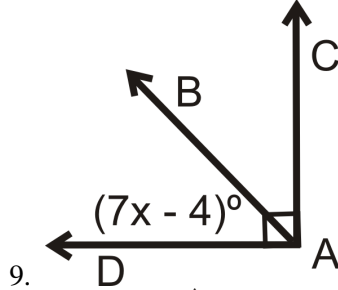
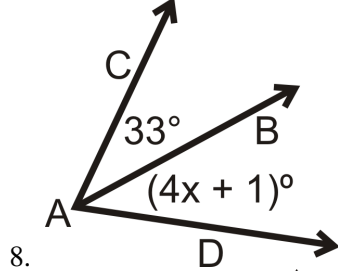
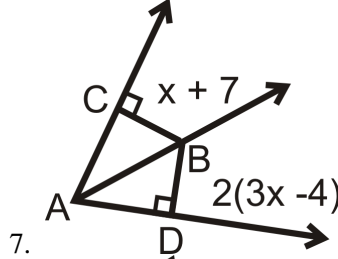
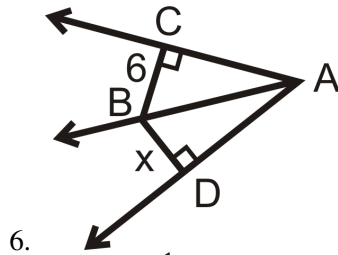
Review Questions

Construction Construct the incenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-2. Draw the inscribed circle too.

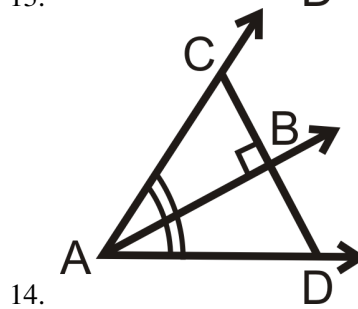
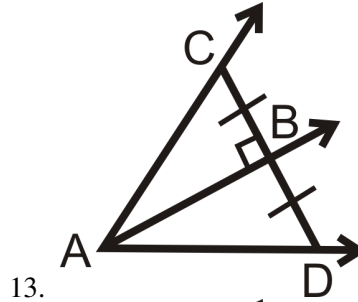
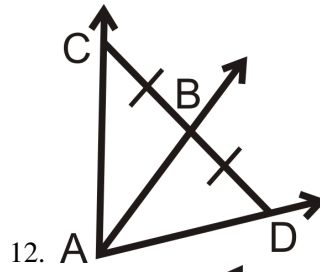


4. Is the incenter always going to be inside of the triangle? Why?
5. For an equilateral triangle, what can you conclude about the circumcenter and the incenter?

For questions 6-11, \overrightarrow{AB} is the angle bisector of $\angle CAD$. Solve for the missing variable.

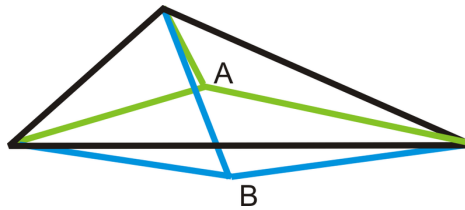


Is there enough information to determine if \overrightarrow{AB} is the angle bisector of $\angle CAD$? Why or why not?

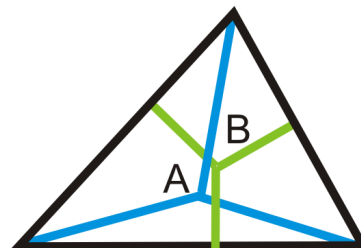


What are points A and B ? How do you know?

15. The blue lines are congruent The green lines are angle bisectors



16. Both sets of lines are congruent The green lines are perpendicular to the sides



17. Fill in the blanks in the Angle Bisector Theorem Converse. Given: $\overline{AD} \cong \overline{DC}$, such that AD and DC are the shortest distances to \overrightarrow{BA} and \overrightarrow{BC} Prove: \overrightarrow{BD} bisects $\angle ABC$

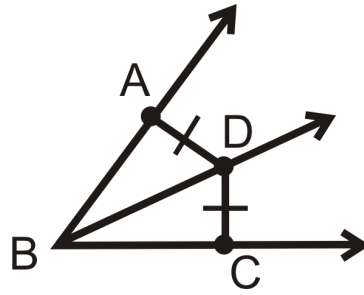


TABLE 5.4:

<i>Statement</i>	<i>Reason</i>
1.	
2.	The shortest distance from a point to a line is perpendicular.
3. $\angle DAB$ and $\angle DCB$ are right angles	
4. $\angle DAB \cong \angle DCB$	
5. $\overline{BD} \cong \overline{BD}$	
6. $\triangle ABD \cong \triangle CBD$	
7.	CPCTC
8. \overrightarrow{BD} bisects $\angle ABC$	

Determine if the following descriptions refer to the incenter or circumcenter of the triangle.

18. A lighthouse on a triangular island is equidistant to the three coastlines.
19. A hospital is equidistant to three cities.
20. A circular walking path passes through three historical landmarks.
21. A circular walking path connects three other straight paths.

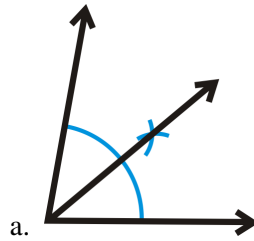
Constructions

22. Construct an equilateral triangle.
23. Construct the angle bisectors of two of the angles to locate the incenter.
24. Construct the perpendicular bisectors of two sides to locate the circumcenter.
25. What do you notice? Use these points to construct an inscribed circle inside the triangle and a circumscribed circle about the triangle.

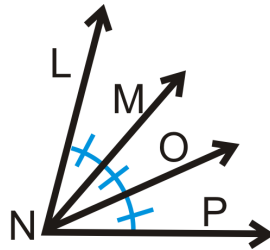
Multi- Step Problem

26. Draw $\angle ABC$ through $A(1, 3)$, $B(3, -1)$ and $C(7, 1)$.
27. Use slopes to show that $\angle ABC$ is a right angle.
28. Use the distance formula to find AB and BC .
29. Construct a line perpendicular to AB through A .
30. Construct a line perpendicular to BC through C .
31. These lines intersect in the interior of $\angle ABC$. Label this point D and draw \overrightarrow{BD} .
32. Is \overrightarrow{BD} the angle bisector of $\angle ABC$? Justify your answer.

Review Queue Answers



b. $\angle LNM \cong \angle MNO \cong \angle ONP$
 $\angle LNO \cong \angle MNP$



a. $5x + 11 = 26$

$$5x = 15$$

$$x = 3$$

b. $9x - 1 = 2(4x + 5)$

$$9x - 1 = 8x + 10$$

$$x = 11^\circ$$

5.4 Medians and Altitudes in Triangles

Learning Objectives

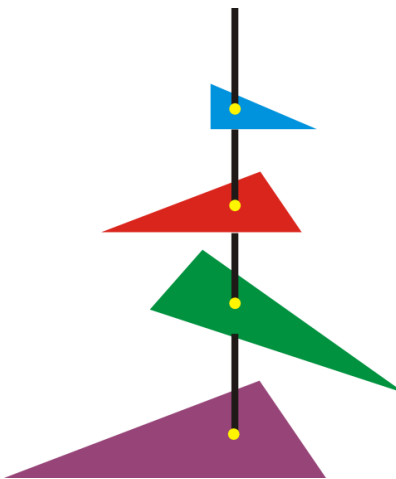
- Define median and find their point of concurrency in a triangle.
- Apply medians to the coordinate plane.
- Construct the altitude of a triangle and find their point of concurrency in a triangle.

Review Queue

- Find the midpoint between (9, -1) and (1, 15).
- Find the equation of the line between the two points from #1.
- Find the equation of the line that is perpendicular to the line from #2 through (-6, 2).

Know What? Triangles are frequently used in art. Your art teacher assigns an art project involving triangles. You decide to make a series of hanging triangles of all different sizes from one long piece of wire. Where should you hang the triangles from so that they balance horizontally?

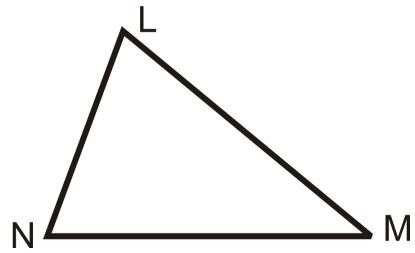
You decide to plot one triangle on the coordinate plane to find the location of this point. The coordinates of the vertices are (0, 0), (6, 12) and (18, 0). What is the coordinate of this point?



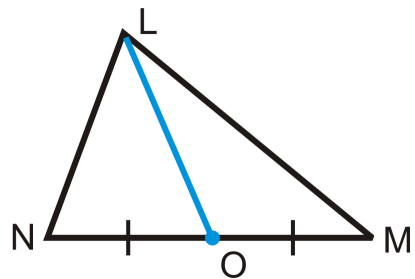
Medians

Median: The line segment that joins a vertex and the midpoint of the opposite side (of a triangle).

Example 1: Draw the median \overline{LO} for $\triangle LMN$ below.



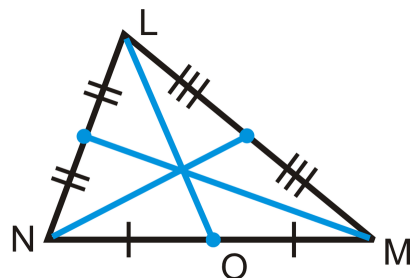
Solution: From the definition, we need to locate the midpoint of \overline{NM} . We were told that the median is \overline{LO} , which means that it will connect the vertex L and the midpoint of \overline{NM} , to be labeled O . Measure NM and make a point halfway between N and M . Then, connect O to L .



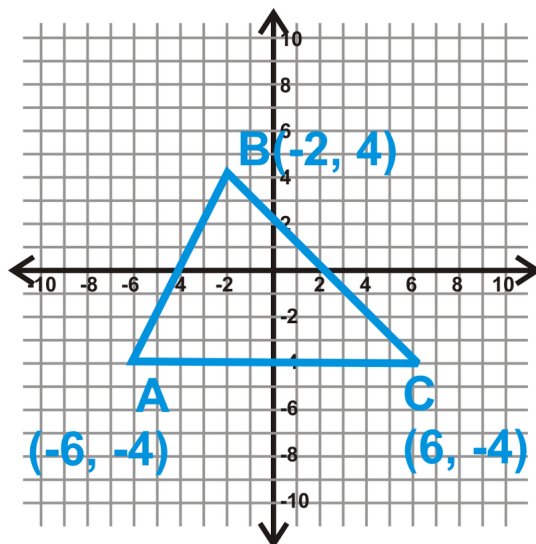
Notice that a median is very different from a perpendicular bisector or an angle bisector. A perpendicular bisector also goes through the midpoint, but it does not necessarily go through the vertex of the opposite side. And, unlike an angle bisector, a median does not necessarily bisect the angle.

Example 2: Find the other two medians of $\triangle LMN$.

Solution: Repeat the process from Example 1 for sides \overline{LN} and \overline{LM} . Be sure to always include the appropriate tick marks to indicate midpoints.



Example 3: Find the equation of the median from B to the midpoint of \overline{AC} for the triangle in the $x - y$ plane below.



Solution: To find the equation of the median, first we need to find the midpoint of \overline{AC} , using the Midpoint Formula.

$$\left(\frac{-6+6}{2}, \frac{-4+(-4)}{2} \right) = \left(\frac{0}{2}, \frac{-8}{2} \right) = (0, -4)$$

Now, we have two points that make a line, B and the midpoint. Find the slope and y -intercept.

$$m = \frac{-4 - 4}{0 - (-2)} = \frac{-8}{2} = -4$$

$$y = -4x + b$$

$$-4 = -4(0) + b$$

$$-4 = b$$

The equation of the median is $y = -4x - 4$

Point of Concurrency for Medians

From Example 2, we saw that the three medians of a triangle intersect at one point, just like the perpendicular bisectors and angle bisectors. This point is called the centroid.

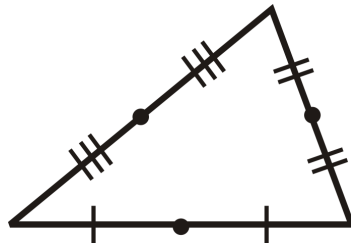
Centroid: The point of concurrency for the medians of a triangle.

Unlike the circumcenter and incenter, the centroid does not have anything to do with circles. It has a different property.

Investigation 5-3: Properties of the Centroid

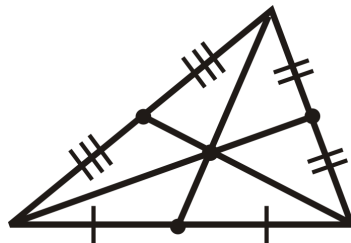
Tools Needed: pencil, paper, ruler, compass

1. Construct a scalene triangle with sides of length 6 cm, 10 cm, and 12 cm (Investigation 4-2). Use the ruler to measure each side and mark the midpoint.

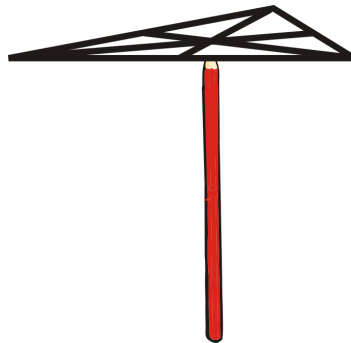


2. Draw in the medians and mark the centroid.

Measure the length of each median. Then, measure the length from each vertex to the centroid and from the centroid to the midpoint. Do you notice anything?



3. Cut out the triangle. Place the centroid on either the tip of the pencil or the pointer of the compass. What happens?



From this investigation, we have discovered the properties of the centroid. They are summarized below.

Concurrency of Medians Theorem: The medians of a triangle intersect in a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side. The centroid is also the “balancing point” of a triangle.

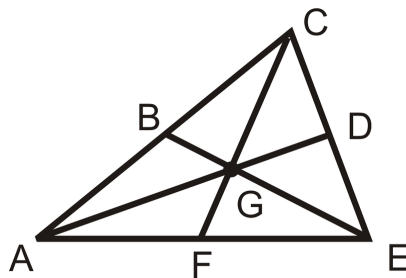
If G is the centroid, then we can conclude:

$$AG = \frac{2}{3}AD, CG = \frac{2}{3}CF, EG = \frac{2}{3}BE$$

$$DG = \frac{1}{3}AD, FG = \frac{1}{3}CF, BG = \frac{1}{3}BE$$

And, combining these equations, we can also conclude:

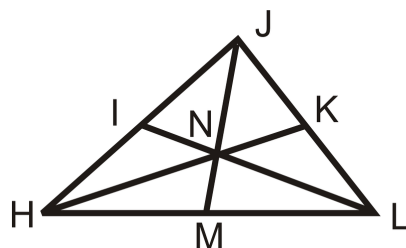
$$DG = \frac{1}{2}AG, FG = \frac{1}{2}CG, BG = \frac{1}{2}EG$$



In addition to these ratios, G is also the balance point of $\triangle ACE$. This means that the triangle will balance when placed on a pencil (#3 in Investigation 5-3) at this point.

Example 4: $I, K,$ and M are midpoints of the sides of $\triangle HJL$.

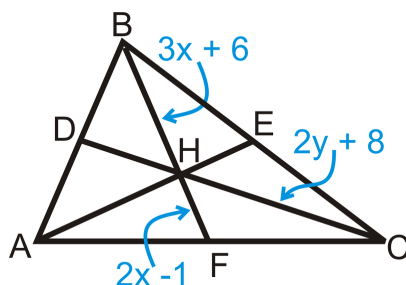
- a) If $JM = 18$, find JN and NM .
- b) If $HN = 14$, find NK and NK .



Solution:

- a) JN is two-thirds of JM . So, $JN = \frac{2}{3} \cdot 18 = 12$. NM is either half of 12, a third of 18 or $18 - 12$. $NM = 6$.
- b) HN is two-thirds of HK . So, $14 = \frac{2}{3} \cdot HK$ and $HK = 14 \cdot \frac{3}{2} = 21$. NK is a third of 21, half of 14, or $21 - 14$. $NK = 7$.

Example 5: Algebra Connection H is the centroid of $\triangle ABC$ and $DC = 5y - 16$. Find x and y .



Solution: HF is half of BH . Use this information to solve for x . For y , HC is two-thirds of DC . Set up an equation for both.

$\frac{1}{2}BH = HF \text{ or } BH = 2HF$ $3x + 6 = 2(2x - 1)$ $3x + 6 = 4x - 2$ $8 = x$	$HC = \frac{2}{3}DC \text{ or } \frac{3}{2}HC = DC$ $\frac{3}{2}(2y + 8) = 5y - 16$ $3y + 12 = 5y - 16$ $28 = 2y$
--	---

Altitudes

The last line segment within a triangle is an altitude. It is also called the height of a triangle.

Altitude: A line segment from a vertex and perpendicular to the opposite side.

Here are a few examples.

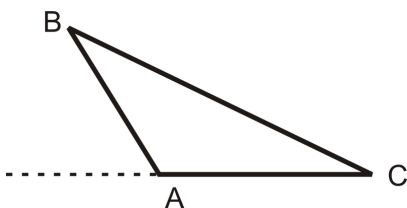


As you can see, an altitude can be a side of a triangle or outside of the triangle. When a triangle is a right triangle, the altitude, or height, is the leg. If the triangle is obtuse, then the altitude will be outside of the triangle. **To construct an altitude, use Investigation 3-2** (constructing a perpendicular line through a point not on the given line). Think of the vertex as the point and the given line as the opposite side.

Investigation 5-4: Constructing an Altitude for an Obtuse Triangle

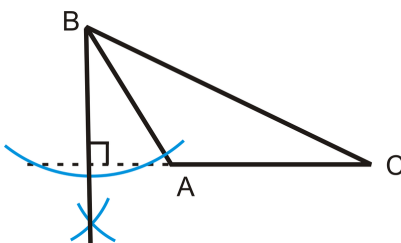
Tools Needed: pencil, paper, compass, ruler

- Draw an obtuse triangle. Label it $\triangle ABC$, like the picture to the right. Extend side \overline{AC} , beyond point A.



- Using Investigation 3-2, construct a perpendicular line to \overline{AC} , through B.

The altitude does not have to extend past side \overline{AC} , as it does in the picture. Technically the height is only the vertical distance from the highest vertex to the opposite side.



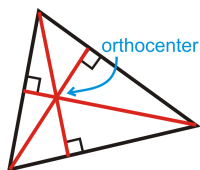
As was true with perpendicular bisectors, angle bisectors, and medians, the altitudes of a triangle are also concurrent. Unlike the other three, the point does not have any special properties.

Orthocenter: The point of concurrency for the altitudes of triangle.

Here is what the orthocenter looks like for the three triangles. It has three different locations, much like the perpendicular bisectors.

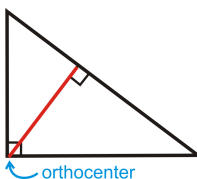
TABLE 5.5:

Acute Triangle



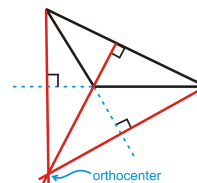
The orthocenter is inside the triangle.

Right Triangle



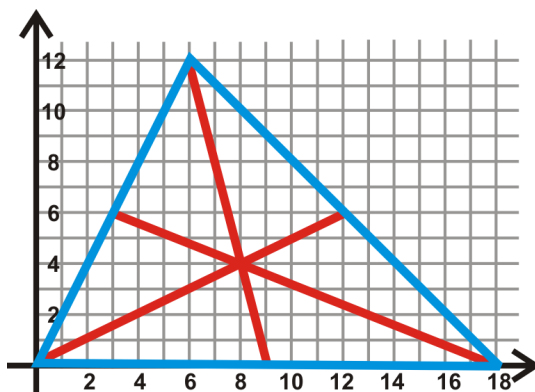
The legs of the triangle are two of the altitudes. The orthocenter is the vertex of the right angle.

Obtuse Triangle



The orthocenter is outside the triangle.

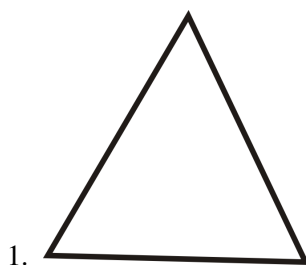
Know What? Revisited The point that you should put the wire through is the centroid. That way, each triangle will balance on the wire.

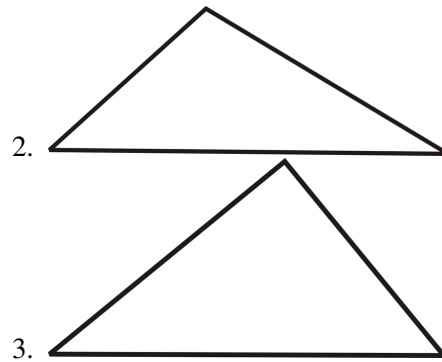


The triangle that we wanted to plot on the $x - y$ plane is to the right. Drawing all the medians, it looks like the centroid is $(8, 4)$. To verify this, you could find the equation of two medians and set them equal to each other and solve for x . Two equations are $y = \frac{1}{2}x$ and $y = -4x + 36$. Setting them equal to each other, we find that $x = 8$ and then $y = 4$.

Review Questions

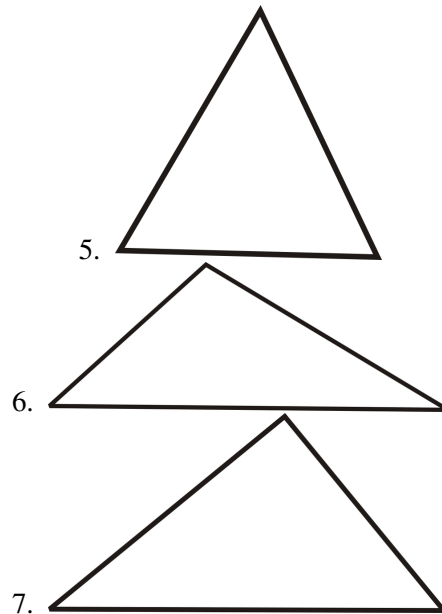
Construction Construct the centroid for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-3.





4. Is the centroid always going to be inside of the triangle? Why?

Construction Construct the orthocenter for the following triangles by tracing each triangle onto a piece of paper and using Investigations 3-2 and 5-4.

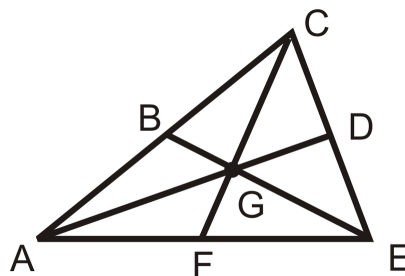


8. What do you think will happen if the triangle is equilateral? What can we say about the incenter, circumcenter, centroid, and orthocenter? Why do you think this is?
9. How many lines do you actually have to “construct” to find any point of concurrency?

For questions 10-13, find the equation of each median, from vertex A to the opposite side, \overline{BC} .

10. $A(9, 5), B(2, 5), C(4, 1)$
 11. $A(-2, 3), B(-3, -7), C(5, -5)$
 12. $A(-1, 5), B(0, -1), C(6, 3)$
 13. $A(6, -3), B(-5, -4), C(-1, -8)$

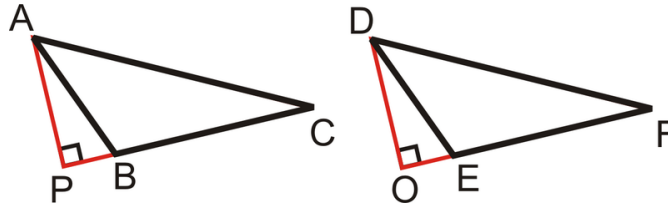
For questions 14-18, $B, D,$ and F are the midpoints of each side and G is the centroid. Find the following lengths.



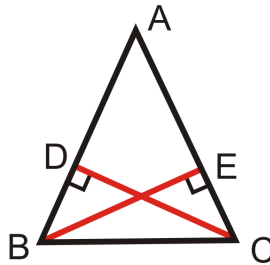
14. If $BG = 5$, find GE and BE
15. If $CG = 16$, find GF and CF
16. If $AD = 30$, find AG and GD
17. If $GF = x$, find GC and CF
18. If $AG = 9x$ and $GD = 5x - 1$, find x and AD .

Write a two-column proof.

19. Given: $\triangle ABC \cong \triangle DEF$ \overline{AP} and \overline{DO} are altitudes Prove: $\overline{AP} \cong \overline{DO}$



20. Given: Isosceles $\triangle ABC$ with legs \overline{AB} and \overline{AC} $\overline{BD} \perp \overline{DC}$ and $\overline{CE} \perp \overline{BE}$ Prove: $\overline{BD} \cong \overline{CE}$



Use $\triangle ABC$ with $A(-2, 9)$, $B(6, 1)$ and $C(-4, -7)$ for questions 21-26.

21. Find the midpoint of \overline{AB} and label it M .
22. Write the equation of \overrightarrow{CM} .
23. Find the midpoint of \overline{BC} and label it N .
24. Write the equation of \overrightarrow{AN} .
25. Find the intersection of \overrightarrow{CM} and \overrightarrow{AN} .
26. What is this point called?

Another way to find the centroid of a triangle in the coordinate plane is to find the midpoint of one side and then find the point two thirds of the way from the third vertex to this point. To find the point two thirds of the way from point $A(x_1, y_1)$ to $B(x_2, y_2)$ use the formula: $\left(\frac{x_1+2x_2}{3}, \frac{y_1+2y_2}{3}\right)$. Use this method to find the centroid in the following problems.

27. $(-1, 3)$, $(5, -2)$ and $(-1, -4)$
28. $(1, -2)$, $(-5, 4)$ and $(7, 7)$
29. Use the coordinates (x_1, y_1) , (x_2, y_2) and (x_3, y_3) and the method used in the last two problems to find a formula for the centroid of a triangle in the coordinate plane.
30. Use your formula from problem 29 to find the centroid of the triangle with vertices $(2, -7)$, $(-5, 1)$ and $(6, -9)$.

Review Queue Answers

a. $midpoint = \left(\frac{9+1}{2}, \frac{-1+15}{2}\right) = (5, 7)$

$$\text{b. } m = \frac{15+1}{1-9} = \frac{16}{-8} = -2 \qquad 15 = -2(1) + b \qquad y = -2x + 17$$

$$17 = b$$

$$\begin{aligned} \text{c. } y &= \frac{1}{2}x + b \\ 2 &= \frac{1}{2}(-6) + b \\ 2 &= -3 + b \\ 5 &= b \\ y &= \frac{1}{2}x + 5 \end{aligned}$$

5.5 Inequalities in Triangles

Learning Objectives

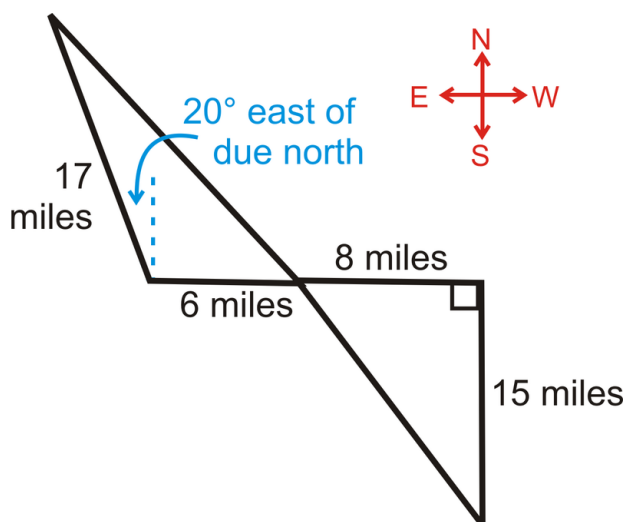
- Determine relationships among the angles and sides of a triangle.
- Understand the Triangle Inequality Theorem.
- Understand the Hinge Theorem and its converse.

Review Queue

Solve the following inequalities.

- $4x - 9 \leq 19$
- $-5 > -2x + 13$
- $\frac{2}{3}x + 1 \geq 13$
- $-7 < 3x - 1 < 14$

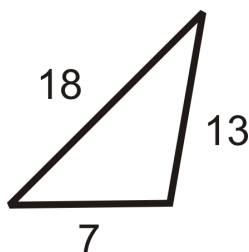
Know What? Two mountain bike riders leave from the same parking lot and head in opposite directions, on two different trails. The first rider goes 8 miles due west, then rides due south for 15 miles. The second rider goes 6 miles due east, then changes direction and rides 20° east of due north for 17 miles. Both riders have been travelling for 23 miles, but which one is further from the parking lot?



Comparing Angles and Sides

Look at the triangle to the right. The sides of the triangle are given. Can you determine which angle is the largest?

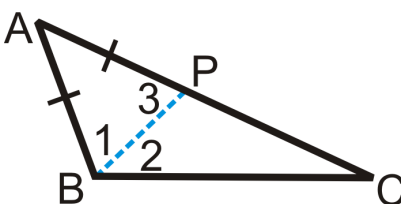
As you might guess, the largest angle will be opposite 18 because it is the longest side. Similarly, the smallest angle will be opposite the shortest side, 7. Therefore, the angle measure in the middle will be opposite 13.



Theorem 5-9: If one side of a triangle is longer than another side, then the angle opposite the longer side will be larger than the angle opposite the shorter side.

Converse of Theorem 5-9: If one angle in a triangle is larger than another angle in a triangle, then the side opposite the larger angle will be longer than the side opposite the smaller angle.

Proof of Theorem 5-9



Given: $AC > AB$

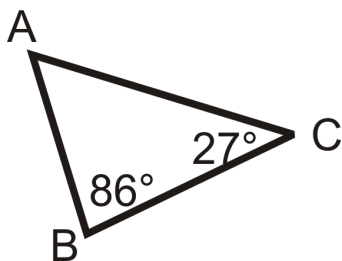
Prove: $m\angle ABC > m\angle C$

TABLE 5.6:

<i>Statement</i>	<i>Reason</i>
1. $AC > AB$	Given
2. Locate point P such that $AB = AP$	Ruler Postulate
3. $\triangle ABP$ is an isosceles triangle	Definition of an isosceles triangle
4. $m\angle 1 = m\angle 3$	Base Angles Theorem
5. $m\angle 3 = m\angle 2 + m\angle C$	Exterior Angle Theorem
6. $m\angle 1 = m\angle 2 + m\angle C$	Substitution PoE
7. $m\angle ABC = m\angle 1 + m\angle 2$	Angle Addition Postulate
8. $m\angle ABC = m\angle 2 + m\angle 2 + m\angle C$	Substitution PoE
9. $m\angle ABC > m\angle C$	Definition of “greater than” (from step 8)

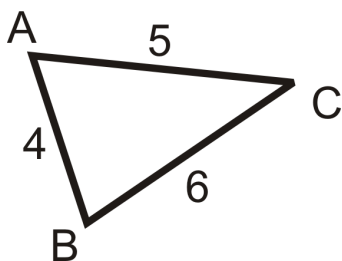
To prove the converse, we will need to do so indirectly. This will be done in the extension at the end of this chapter.

Example 1: List the sides in order, from shortest to longest.



Solution: First, we need to find $m\angle A$. From the Triangle Sum Theorem, $m\angle A + 86^\circ + 27^\circ = 180^\circ$. So, $m\angle A = 67^\circ$. From Theorem 5-9, we can conclude that the longest side is opposite the largest angle. 86° is the largest angle, so AC is the longest side. The next largest angle is 67° , so BC would be the next longest side. 27° is the smallest angle, so AB is the shortest side. In order from shortest to longest, the answer is: AB, BC, AC .

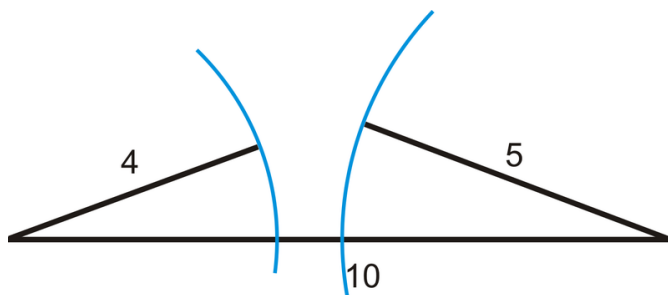
Example 2: List the angles in order, from largest to smallest.



Solution: Just like with the sides, the largest angle is opposite the longest side. The longest side is BC , so the largest angle is $\angle A$. Next would be $\angle B$ and finally $\angle C$ is the smallest angle.

Triangle Inequality Theorem

Can any three lengths make a triangle? The answer is no. There are limits on what the lengths can be. For example, the lengths 1, 2, 3 cannot make a triangle because $1 + 2 = 3$, so they would all lie on the same line. The lengths 4, 5, 10 also cannot make a triangle because $4 + 5 = 9$.



The arc marks show that the two sides would never meet to form a triangle.

Triangle Inequality Theorem: The sum of the lengths of any two sides of a triangle must be greater than the length of the third.

Example 3: Do the lengths below make a triangle?

a) 4.1, 3.5, 7.5

b) 4, 4, 8

c) 6, 7, 8

Solution: Even though the Triangle Inequality Theorem says “the sum of the length of any two sides,” really, it is referring to the sum of the lengths of the two shorter sides must be longer than the third.

a) $4.1 + 3.5 > 7.5$ Yes, these lengths could make a triangle.

b) $4 + 4 = 8$ No, not a triangle. Two lengths cannot equal the third.

c) $6 + 7 > 8$ Yes, these lengths could make a triangle.

Example 4: Find the possible lengths of the third side of a triangle if the other two sides are 10 and 6.

Solution: The Triangle Inequality Theorem can also help you determine the possible range of the third side of a triangle. The two given sides are 6 and 10, so the third side, s , can either be the shortest side or the longest side. For example s could be 5 because $6 + 5 > 10$. It could also be 15 because $6 + 10 > 15$. Therefore, we write the possible values of s as a range, $4 < s < 16$.

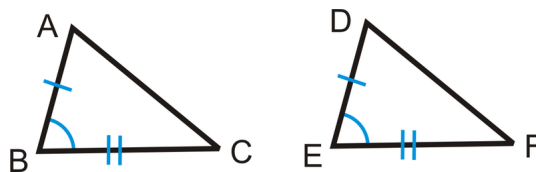


Notice the range is no less than 4, and not equal to 4. The third side could be 4.1 because $4.1 + 6$ would be greater than the third side, 10. For the same reason, s cannot be greater than 16, but it could 15.9. In this case, s would be the longest side and $10 + 6$ must be greater than s to form a triangle.

If two sides are lengths a and b , then the third side, s , has the range $a - b < s < a + b$.

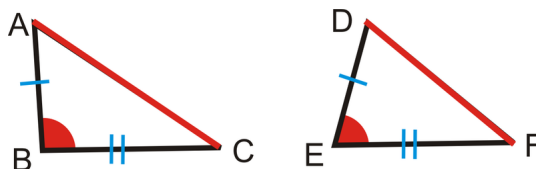
The SAS Inequality Theorem (also called the Hinge Theorem)

The Hinge Theorem is an extension of the Triangle Inequality Theorem using two triangles. If we have two congruent triangles $\triangle ABC$ and $\triangle DEF$, marked below:



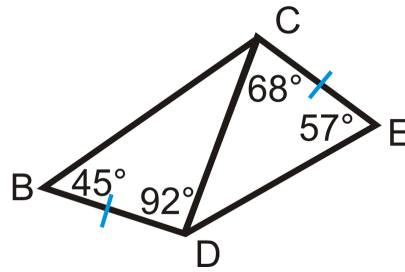
Therefore, if $AB = DE$ and $BC = EF$ and $m\angle B > m\angle E$, then $AC > DF$.

Now, let's adjust $m\angle B > m\angle E$. Would that make $AC > DF$? Yes. See the picture below.



The SAS Inequality Theorem (Hinge Theorem): If two sides of a triangle are congruent to two sides of another triangle, but the included angle of one triangle has greater measure than the included angle of the other triangle, then the third side of the first triangle is longer than the third side of the second triangle.

Example 5: List the sides in order, from least to greatest.



Solution: Let's start with $\triangle DCE$. The missing angle is 55° . By Theorem 5-9, the sides, in order are CE, CD , and DE .

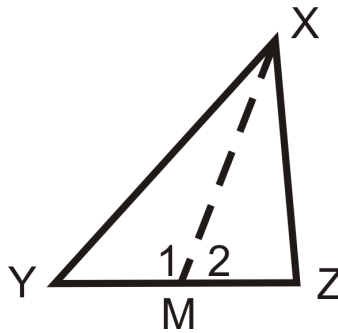
For $\triangle BCD$, the missing angle is 43° . Again, by Theorem 5-9, the order of the sides is BD, CD , and BC .

By the SAS Inequality Theorem, we know that $BC > DE$, so the order of all the sides would be: $BD = CE, CD, DE, BC$.

SSS Inequality Theorem (also called the Converse of the Hinge Theorem)

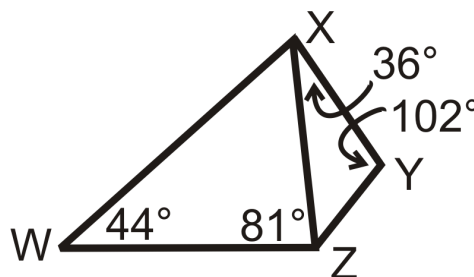
SSS Inequality Theorem: If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.

Example 6: If \overline{XM} is a median of $\triangle XYZ$ and $XY > XZ$, what can we say about $m\angle 1$ and $m\angle 2$? What we can deduce from the following diagrams.



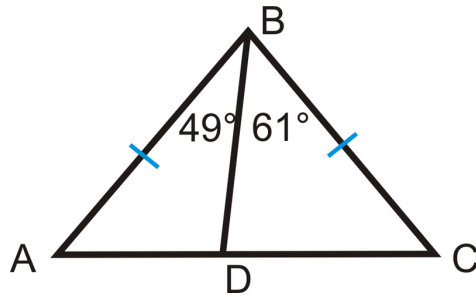
Solution: By the definition of a median, M is the midpoint of \overline{YZ} . This means that $YM = MZ$. $MX = MX$ by the Reflexive Property and we know that $XY > XZ$. Therefore, we can use the SSS Inequality Theorem to conclude that $m\angle 1 > m\angle 2$.

Example 7: List the sides of the two triangles in order, from least to greatest.



Solution: Here we have no congruent sides or angles. So, let's look at each triangle separately. Start with $\triangle XYZ$. First the missing angle is 42° . By Theorem 5-9, the order of the sides is YZ, XY , and XZ . For $\triangle WXZ$, the missing angle is 55° . The order of these sides is XZ, WZ , and WX . Because the longest side in $\triangle XYZ$ is the shortest side in $\triangle WXZ$, we can put all the sides together in one list: YZ, XY, XZ, WZ, WX .

Example 8: Below is isosceles triangle $\triangle ABC$. List everything you can about the triangle and why.

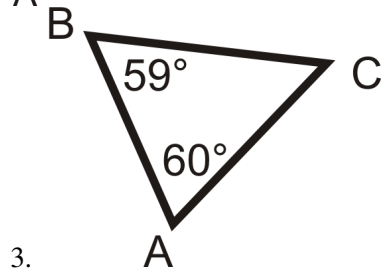
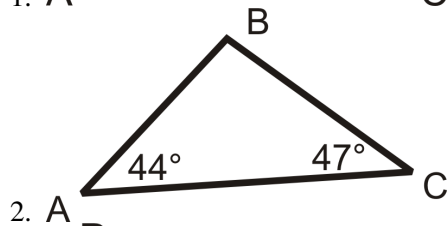
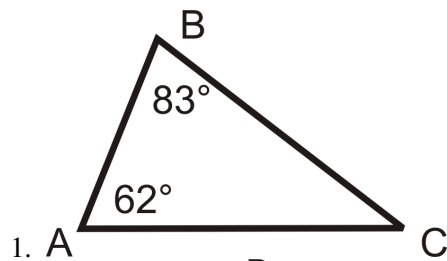
**Solution:**

- $AB = BC$ because it is given.
- $m\angle A = m\angle C$ by the Base Angle Theorem.
- $AD < DC$ because $m\angle ABD < m\angle CBD$ and the SAS Triangle Inequality Theorem.

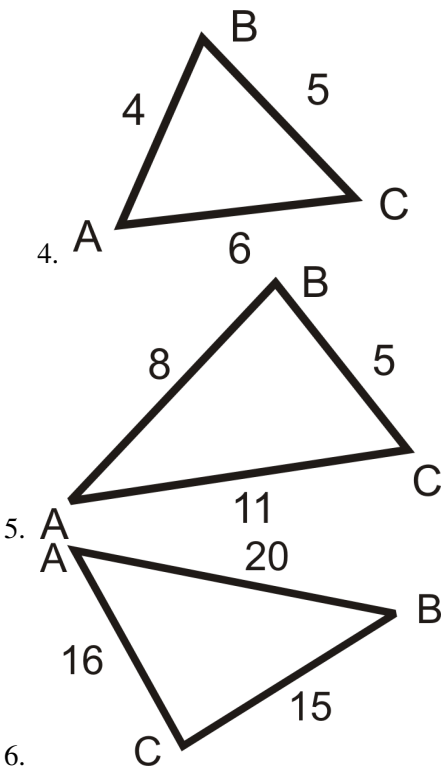
Know What? Revisited Even though the two sets of lengths are not equal, they both add up to 23. Therefore, the second rider is further away from the parking lot because $110^\circ > 90^\circ$.

Review Questions

For questions 1-3, list the sides in order from shortest to longest.



For questions 4-6, list the angles from largest to smallest.

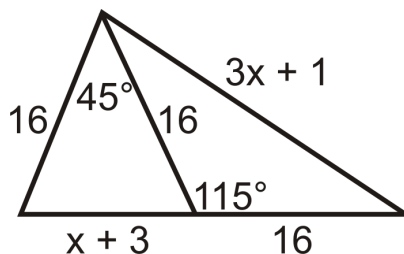


Determine if the sets of lengths below can make a triangle. If not, state why.

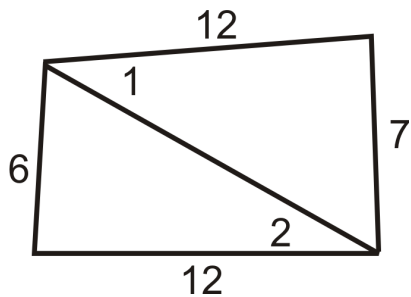
- 7. 6, 6, 13
- 8. 1, 2, 3
- 9. 7, 8, 10
- 10. 5, 4, 3
- 11. 23, 56, 85
- 12. 30, 40, 50

If two lengths of the sides of a triangle are given, determine the range of the length of the third side.

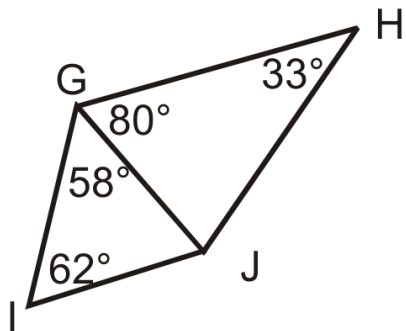
- 13. 8 and 9
- 14. 4 and 15
- 15. 20 and 32
- 16. The base of an isosceles triangle has length 24. What can you say about the length of each leg?
- 17. What conclusions can you draw about x ?



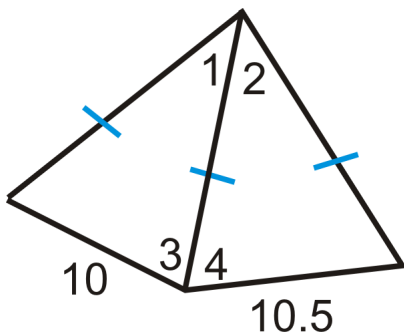
- 18. Compare $m\angle 1$ and $m\angle 2$.



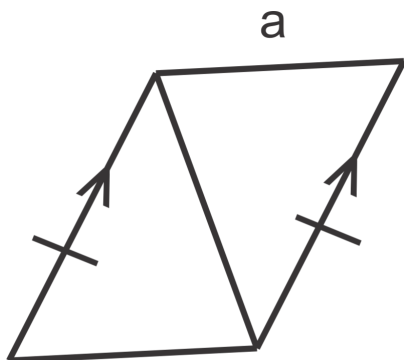
19. List the sides from shortest to longest.



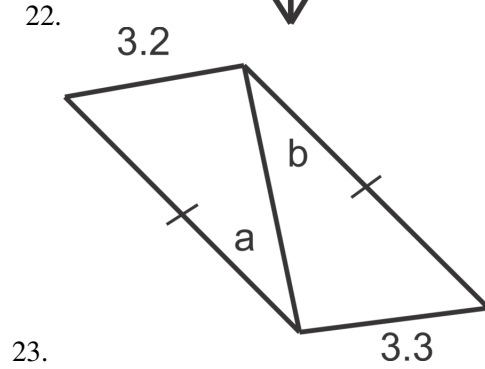
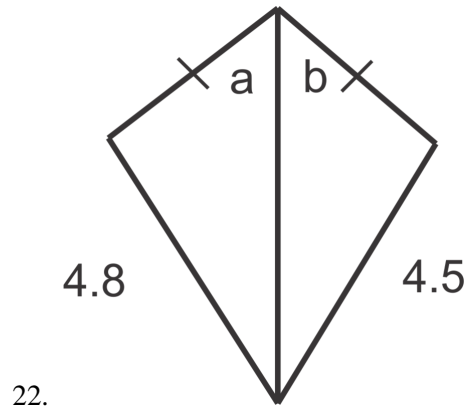
20. Compare $m\angle 1$ and $m\angle 2$. What can you say about $m\angle 3$ and $m\angle 4$?



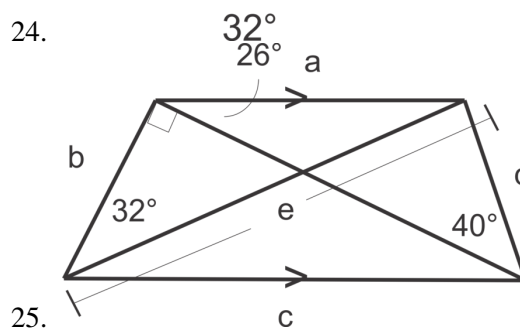
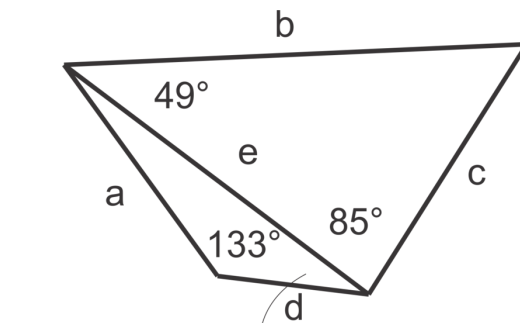
In questions 21-23, compare the measures of a and b .



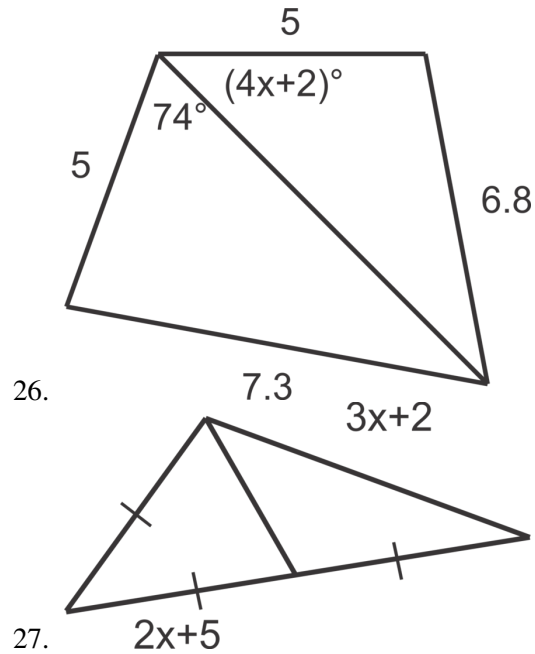
21. b



In questions 24 and 25, list the measures of the sides in order from least to greatest

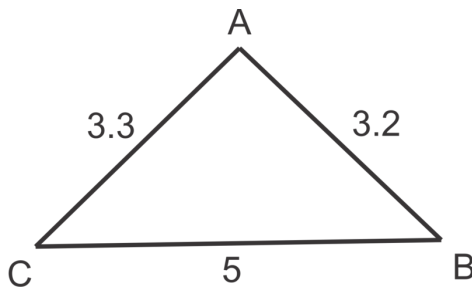


In questions 26 and 27 determine the range of possible values for x .

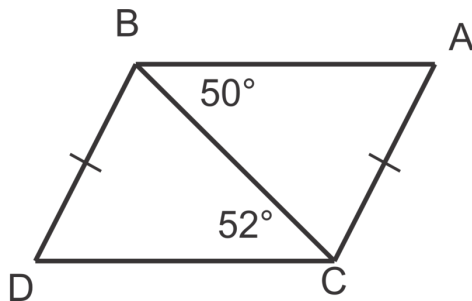


In questions 28 and 29 explain why the conclusion is false.

28. Conclusion: $m\angle C < m\angle B < m\angle A$



29. Conclusion: $AB < DC$



30. If \overline{AB} is a median of $\triangle CAT$ and $CA > AT$, explain why $\angle ABT$ is acute. You may wish to draw a diagram.

Review Queue Answers

- a. $4x - 9 \leq 19$
 $4x \leq 28$
 $x \leq 7$

b. $-5 > -2x + 13$

$-18 > -2x$

$9 < x$

c. $\frac{2}{3}x + 1 \geq 13$

$\frac{2}{3}x \geq 12$

$x \geq 18$

d. $-7 < 3x - 1 < 14$

$-6 < 3x < 15$

$-2 < x < 5$

5.6 Extension: Indirect Proof

The indirect proof or proof by contradiction is a part of 41 out of 50 states' mathematic standards. Depending on the state, the teacher may choose to use none, part or all of this section.

Learning Objectives

- Reason indirectly to develop proofs.

Until now, we have proved theorems true by direct reasoning, where conclusions are drawn from a series of facts and previously proven theorems. However, we cannot always use direct reasoning to prove every theorem.

Indirect Proof: When the conclusion from a hypothesis is assumed false (or opposite of what it states) and then a contradiction is reached from the given or deduced statements.

The easiest way to understand indirect proofs is by example. You may choose to use the two-column format or a paragraph proof. First we will explore indirect proofs with algebra and then geometry.

Indirect Proofs in Algebra

Example 1: If $x = 2$, then $3x - 5 \neq 10$. Prove this statement is true by contradiction.

Solution: In an indirect proof the first thing you do is assume the conclusion of the statement is false. In this case, we will assume the opposite of $3x - 5 \neq 10$

If $x = 2$, then $3x - 5 = 10$

Now, proceed with this statement, as if it is true. Solve for x .

$$3x - 5 = 10$$

$$3x = 15$$

$$x = 5$$

$x = 5$ contradicts the given statement that $x = 2$. Hence, our assumption is incorrect and $3x - 5$ cannot equal 10.

Example 2: If n is an integer and n^2 is odd, then n is odd. Prove this is true indirectly.

Solution: First, assume the opposite of “ n is odd.”

n is even.

Now, square n and see what happens.

If n is even, then $n = 2a$, where a is any integer.

$$n^2 = (2a)^2 = 4a^2$$

This means that n^2 is a multiple of 4. No odd number can be divided evenly by an even number, so this contradicts our assumption that n is even. Therefore, n must be odd if n^2 is odd.

Indirect Proofs in Geometry

Example 3: If $\triangle ABC$ is isosceles, then the measure of the base angles cannot be 92° . Prove this indirectly.

Solution: Assume the opposite of the conclusion.

The measure of the base angles is 92 $^\circ$.

If the base angles are 92° , then they add up to 184° . This contradicts the Triangle Sum Theorem that says all triangles add up to 180° . Therefore, the base angles cannot be 92° .

Example 4: Prove the SSS Inequality Theorem is true by contradiction.

Solution: The SSS Inequality Theorem says: “If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.” First, assume the opposite of the conclusion.

The included angle of the first triangle is less than or equal to the included angle of the second triangle.

If the included angles are equal then the two triangles would be congruent by SAS and the third sides would be congruent by CPCTC. This contradicts the hypothesis of the original statement “the third side of the first triangle is longer than the third side of the second.” Therefore, the included angle of the first triangle must be larger than the included angle of the second.

To summarize:

- Assume the **opposite** of the conclusion (second half) of the statement.
- Proceed as if this assumption is true to find the **contradiction**.
- Once there is a contradiction, the original statement is true.
- **DO NOT use specific examples.** Use variables so that the contradiction can be generalized.

Review Questions

Prove the following statements true indirectly.

1. If n is an integer and n^2 is even, then n is even.
2. If $m\angle A \neq m\angle B$ in $\triangle ABC$, then $\triangle ABC$ is not equilateral.
3. If $x > 3$, then $x^2 > 9$.
4. The base angles of an isosceles triangle are congruent.
5. If x is even and y is odd, then $x + y$ is odd.
6. In $\triangle ABE$, if $\angle A$ is a right angle, then $\angle B$ cannot be obtuse.
7. If A, B , and C are collinear, then $AB + BC = AC$ (Segment Addition Postulate).
8. If a collection of nickels and dimes is worth 85 cents, then there must be an odd number of nickels.
9. Hugo is taking a true/false test in his Geometry class. There are five questions on the quiz. The teacher gives her students the following clues: The last answer on the quiz is not the same as the fourth answer. The third answer is true. If the fourth answer is true, then the one before it is false. Use an indirect proof to prove that the last answer on the quiz is true.

10. On a test of 15 questions, Charlie claims that his friend Suzie must have gotten at least 10 questions right. Another friend, Larry, does not agree and suggests that Suzie could not have gotten that many correct. Rebecca claims that Suzie certainly got at least one question correct. If *only one* of these statements is true, how many questions did Suzie get right?

5.7 Chapter 5 Review

Keywords, Theorems and Postulates

Midsegment

A line segment that connects two midpoints of adjacent sides of a triangle.

Midsegment Theorem

The midsegment of a triangle is half the length of the side it is parallel to.

Perpendicular Bisector Theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Perpendicular Bisector Theorem Converse

If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Point of Concurrency

When three or more lines intersect at the same point.

Circumcenter

The point of concurrency for the perpendicular bisectors of the sides of a triangle.

Concurrency of Perpendicular Bisectors Theorem

The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the vertices.

Angle Bisector Theorem

If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

Angle Bisector Theorem Converse

If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

Incenter

The point of concurrency for the angle bisectors of a triangle.

Concurrency of Angle Bisectors Theorem

The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

Median

The line segment that joins a vertex and the midpoint of the opposite side (of a triangle).

Centroid

The point of concurrency for the medians of a triangle.

Concurrency of Medians Theorem

The medians of a triangle intersect in a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side.

Altitude

A line segment from a vertex and perpendicular to the opposite side.

Orthocenter

The point of concurrency for the altitudes of triangle.

Theorem 5-9

If one side of a triangle is longer than another side, then the angle opposite the longer side will be larger than the angle opposite the shorter side.

Converse of Theorem 5-9

If one angle in a triangle is larger than another angle in a triangle, then the side opposite the larger angle will be longer than the side opposite the smaller angle.

Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle must be greater than the length of the third.

SAS Inequality Theorem

If two sides of a triangle are congruent to two sides of another triangle, but the included angle of one triangle has greater measure than the included angle of the other triangle, then the third side of the first triangle is longer than the third side of the second triangle.

SSS Inequality Theorem

If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.

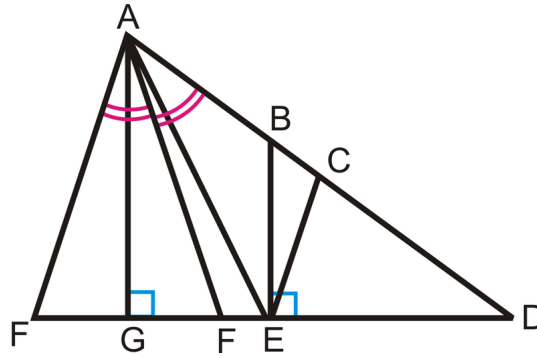
Indirect Proof

When the conclusion from a hypothesis is assumed false (or opposite of what it states) and then a contradiction is reached from the given or deduced statements.

Review

If C and E are the midpoints of the sides they lie on, find:

1. The perpendicular bisector of \overline{FD} .
2. The median of \overline{FD} .
3. The angle bisector of $\angle FAD$.
4. A midsegment.
5. An altitude.



6. Trace $\triangle FAD$ onto a piece of paper with the perpendicular bisector. Construct another perpendicular bisector. What is the point of concurrency called? Use this information to draw the appropriate circle.
7. Trace $\triangle FAD$ onto a piece of paper with the angle bisector. Construct another angle bisector. What is the point of concurrency called? Use this information to draw the appropriate circle.
8. Trace $\triangle FAD$ onto a piece of paper with the median. Construct another median. What is the point of concurrency called? What are its properties?
9. Trace $\triangle FAD$ onto a piece of paper with the altitude. Construct another altitude. What is the point of concurrency called? Which points of concurrency can lie outside a triangle?
10. A triangle has sides with length $x + 6$ and $2x - 1$. Find the range of the third side.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9690> .